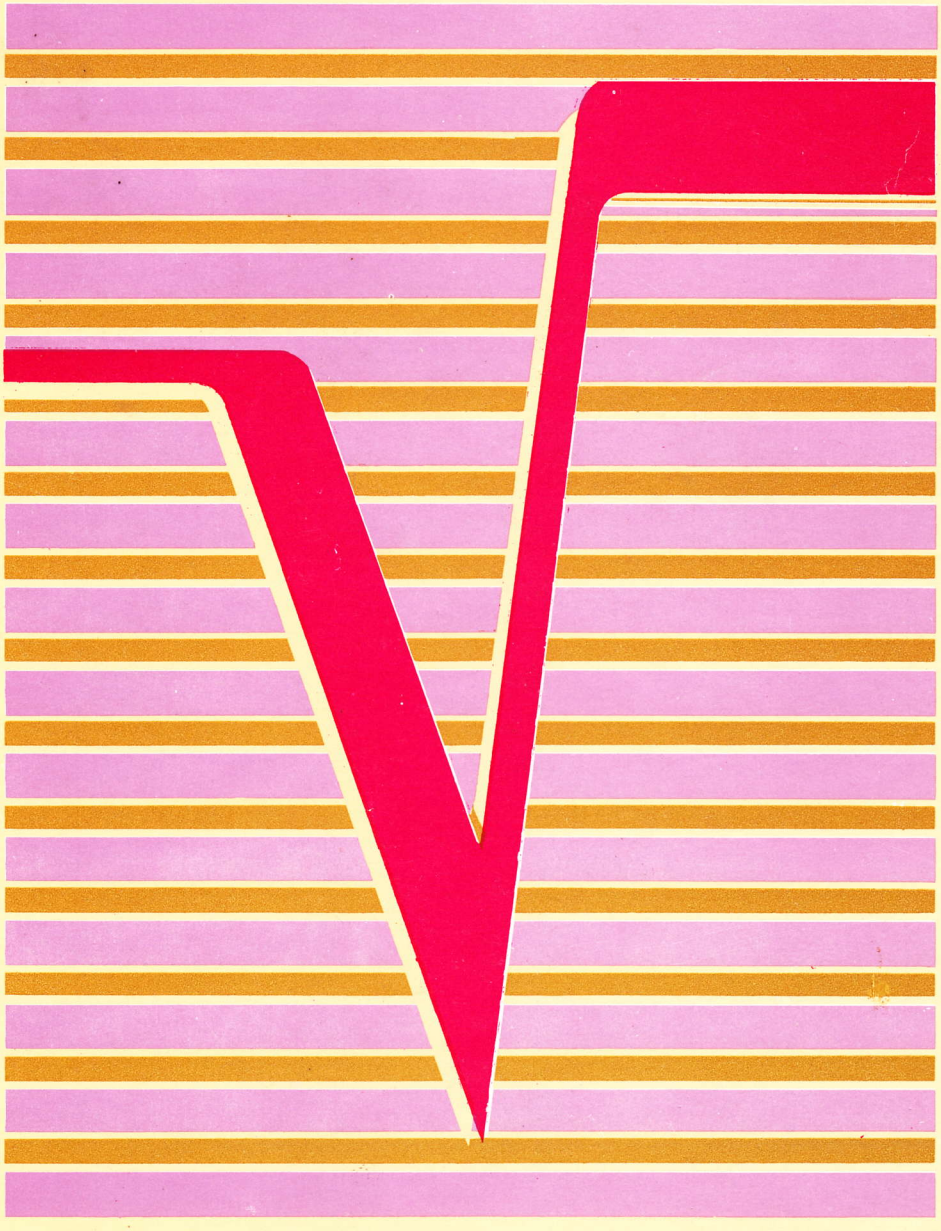
DRS. WARDIMAN

HITUNG INTEGRAL



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Cetakan Ketiga , Juli 1987

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Isi diluar tanggung jawab percetakan.

□ KATA PENGANTAR

Mengingat akan kesukaran-kesukaran yang dialami oleh mahasiswa-mahasiswa dalam menyelesaikan soal-soal mathematika, terutama dalam hitung integral, maka kami susunlah suatu metode untuk menyelesaikan soal-soal hitung integral, yang sekira dapat menolong mahasiswa-mahasiswa dan pelajar-pelajar dalam mengatasi kesukaran-kesukaran tersebut.

Buku ini selain untuk mahasiswa-mahasiswa eksakta, dapat juga digunakan untuk mahasiswa-mahasiswa non eksakta atau pelajar-pelajar Sekolah Teknik Menengah, yang membutuhkan pelajaran mathematika.

Dalam buku ini titik berat kami letakkan pada ketrampilan menyelesaikan soal-soal. Oleh karena itu, di samping teori-teori, atau metode-metode diberikan juga contoh-contoh penyelesaian soal-soal, dari yang sederhana sampai yang sukar, dan yang sesuai dengan metode-metode tersebut.

Akhirnya, kami ucapkan banyak-banyak terima kasih kepada pembaca, yang sudi memberikan kritik-kritik dan petunjuk pembetulan kesalahan-kesalahan yang diberikan dengan ikhlas terutama untuk membangun perbaikan buku ini.

Sekian.

Yogyakarta, Januari 1982
Penulis
Drs. Wardiman

□ DAFTAR ISI

halaman

KATA PENGANTAR i

DAFTAR ISI iii

BAB I RUMUS–RUMUS DASAR INTEGRAL 1

1. Integral tak tertentu 1
2. Rumus–rumus integrasi 1

Soal–soal 15

3. Integrasi Persiil 22

Soal–soal 28

BAB II METODE INTEGRASI 30

1. Integral dari bentuk fungsi goniometri 30

a. Integral dari bentuk :

$\sin^m x \cos^n x \, dx$ 30

b. Integral dari bentuk :

$tg^m x \sec^n x \, dx$ dan $ctg^m x \operatorname{cosec}^n x \, dx$ 35

c. Integral dari bentuk : 36

$\int \sin mx \cos nx \, dx$

$\int \sin mx \sin nx \, dx$

$\int \cos mx \cos nx \, dx$

2. Integral dengan menggunakan substitusi 39

a. Substitusi fungsi aljabar 40

b. Substitusi dengan trigonometri 46

c. Integral dari fungsi pecah rasional 49

d. Integral dari fungsi-fungsi irrasional 61

e. Fungsi rasional dari sin x dan cos x 70

Soal–Soal 73

BAB III INTEGRASI FUNGSI HIPERBOLIK 87

1. Rumus–rumus dasar fungsi hiperbolik 87
2. Rumus–rumus dasar integral dari fungsi hiperbolik 88
3. Substitusi fungsi hiperbolik 91

Soal–Soal 94

BAB IV RUMUS–RUMUS REDUKSI 97

1. Rumus reduksi untuk Binomial Differensial 97

Soal–Soal 100

2. Rumus reduksi untuk trigonometrik differensial 102

Soal–Soal 105

BAB V PENYELESAIAN SOAL–SOAL UJIAN 106

DAFTAR PUSTAKA 160

□ BAB I
RUMUS-RUMUS DASAR INTEGRAL

1. INTEGRAL TAK TERTENTU (INDIFINITE INTEGRAL)

Bila F (x) suatu fungsi yang mempunyai derivative F' (x) = f (x), maka F (x) disebut anti - derivative atau integral tak tertentu (indefinite integral) dari f (x), sedang f (x) disebut integran. Integral tak tertentu dari suatu fungsi yang diketahui tidak tunggal; sebagai contoh :, x^3, $x^3 + 7$, $x^3 - 9$ adalah integral tak tertentu dari f (x) = $3x^2$, oleh-karena $\frac{d}{dx}(x^3) = \frac{d}{dx}(x^3 + 7) = \frac{d}{dx}(x^3 - 9) = 3x^2$. Maka integral tak tertentu dari f (x) = $3x^2$ dapat ditulis secara umum $x^3 + C$, dimana C disebut konstante integrasi (constant of integration) yang sembarang.
Untuk menyatakan integral tak tertentu dari f (x) ditulis dengan bentuk :

$$\int f(x)\,dx$$

$$\int 3x^2\,dx = x^3 + C.$$

Jadi :

2. RUMUS-RUMUS INTEGRASI

1. $\int \frac{d}{dx} f(x)\,dx = f(x) + C$
2. $\int (U \pm V)\,dx = \int U\,dx \pm \int V\,dx$
3. $\int k\,U\,dx = k \int U\,dx$, k setiap konstante.

4. $\int U^m \, dU = \frac{U^{m+1}}{m+1} + C, \quad m \neq -1.$

5. $\int \frac{dU}{U} = \ln |U| + C$

6. $\int a^U \, dU = \frac{a^U}{\ln + a} + C, \quad a > o, \quad a \neq 1.$

7. $\int e^U \, dU = e^U + C.$

8. $\int \sin U \, dU = - \cos U = C.$

9. $\int \cos U \, dU = \sin U + C.$

10. $\int \text{tg } U \, dU = \ln |\sec U| + C.$

11. $\int \text{ctg } U \, dU = \ln |\sin U| + C.$

12. $\int \sec U \, dU = \ln |\sec U + \text{tg } U| + C$

13. $\int \text{cosec } U \, dU = \ln |\text{cosec } U - \text{ctg } U| + C.$

14. $\int \sec^2 U \, dU = \text{tg } U + C.$

15. $\int \text{cosec}^2 U \, dU = - \text{ctg } U + C.$

16. $\int \sec U \, \text{tg } U \, dU = \sec U + C.$

17. $\int \text{cosec } U \, \text{ctg } U \, dU = - \text{cosec } U + C.$

18. $\int \frac{dU}{a^2 + U^2} = \frac{1}{a} \text{ arc tg } \frac{U}{a} + C.$

19. $\int \frac{dU}{\sqrt{a^2 - U^2}} = \text{arc sin } \frac{U}{a} + C.$

20. $\int \frac{dU}{U\sqrt{U^2 - a^2}} = \frac{1}{a} \text{ arc sec } \frac{U}{a} + C$

21. $\int \frac{dU}{U^2 - a^2} = \frac{1}{2a} \ln\left|\frac{U - a}{U + a}\right| + C.$

22. $\int \frac{dU}{\sqrt{U^2 + a^2}} = \ln\left| U + \sqrt{U^2 + a^2}\right| + C$

23. $\int \frac{dU}{\sqrt{U^2 - a^2}} = \ln\left| U + \sqrt{U^2 - a^2}\right| + C.$

24. $\int \sqrt{a^2 - U^2}\, dU = \tfrac{1}{2}U\sqrt{a^2 - U^2} + \tfrac{1}{2}a^2 \text{ arc sin } \frac{U}{a} + C.$

25. $\int \sqrt{U^2 + a^2}\, dU = \tfrac{1}{2}U\sqrt{U^2 + a^2} + \tfrac{1}{2}a^2 \ln\left| U + \sqrt{U^2 + a^2}\right| + C.$

26. $\int \sqrt{U^2 - a^2}\, dU = \tfrac{1}{2}U\sqrt{U^2 - a^2} - \tfrac{1}{2}a^2 \ln\left|U + \sqrt{U^2 - a^2}\right| + C.$

Contoh-contoh penyelesaian soal-soal

1. $\int x^6\, dx = \frac{x^7}{7} + C$

2. $\int \frac{dx}{x^3} = \int x^{-3}\, dx = \frac{x^{-3+1}}{-3+1} + C = -\frac{1}{2x^2} + C.$

3. $\int \sqrt[3]{z}\, dz = \int z^{1/3}\, dz = \frac{z^{4/3}}{4/3} + C = \frac{3}{4} z^{4/3} + C.$

4. $\int \frac{dx}{\sqrt[3]{x^2}} = \int x^{-2/3}\, dx = \frac{x^{1/3}}{1/3} + C = 3x^{1/3} + C.$

5. $\int (2x^2 - 5x + 3)\, dx = \int 2x^2\, dx - \int 5x\, dx + \int 3\, dx$

$= \frac{2}{3} x^3 - \frac{5}{2} x^2 + 3x + C.$

6. $\int (1 - x)\sqrt{x}\, dx = \int x^{1/2}\, dx - \int x^{3/2}\, dx = \frac{2}{3} x^{3/2} - \frac{2}{5} x^{5/2} + C.$

7. $\int (3x + 4)^2\, dx = \int (9x^2 + -24x + 16)\, dx = 3x^3 + 12x^2 + 16x + C.$

8. $\int \frac{x^3 + 5x^2 - 4}{x^2}\, dx = \int (x + 5 - 4x^{-2})\, dx = \frac{1}{2}x^2 + 5x + \frac{4}{x} + C.$

9. Hitunglah : (a). $\int (x^3 + 2)^2\, 3x^2\, dx$; (b). $\int (x^3 + 2)^{\frac{1}{2}}\, x^2\, dx$;

(c). $\int \frac{8x^2\, dx}{(x^3 + 2)^2}$; (d). $\int \frac{x^2\, dx}{\sqrt[4]{x^3 + 2}}$

Penyelesaian :

Untuk mempermudah penyelesaian, dimisalkan : $x^3 + 2 = U$, maka differensial dari U adalah : $dU = 3x^2 dx$.

(a). $\int (x^3 + 2)^2\, 3x^2\, dx = \int U^2\, dU = \frac{1}{3} U^3 + C = \frac{1}{3} (x^3 + 2)^3 + C.$

(b). $\int (x^3 + 2)^{\frac{1}{2}}\, x^2\, dx = \frac{1}{3} \int U^{\frac{1}{2}}\, dU = \frac{2}{9} U^{3/2} + C = \frac{2}{9} (x^3 + 2)^{3/2} + C.$

(c). $\int \frac{8x^2\, dx}{(x^3 + 2)^3} = \frac{8}{3} \int (x^3 + 2)^{-3} \cdot 3x^2\, dx = \frac{8}{3} \int U^{-3}\, dU$

$= \frac{8}{3} (-\frac{1}{2} U^{-2}) + C = \frac{4}{3 (x^3 + 2)^2} + C.$

(d). $\int \frac{x^2\, dx}{\sqrt[4]{x^3 + 2}} = \frac{1}{3} \int (x^3 + 2)^{-1/4} \cdot 3x^2\, dx = \frac{1}{3} \int U^{-1/4}\, dU$

$= \frac{4}{9} (U^{3/4}) + C = \frac{4}{9} (x^3 + 2)^{3/4} + C.$

10. $\int 3x\sqrt{1 - 2x^2}\, dx.$

Penyelesaian : Misalkan $1 - 2x^2 = U$, maka $dU = -4x\, dx$.

$$\int 3x\sqrt{1 - 2x^2}\, dx = -\frac{3}{4}\int U^{\frac{1}{2}}\, dU = -\tfrac{1}{2} U^{3/2} + C.$$

$$= -\tfrac{1}{2}(1 - 2x^2)^{3/2} + C.$$

11. Hitung : $\int \frac{(x + 3)\, dx}{(x^2 + 6x)^{1/3}}$

Penyelesaian : misalkan : $x^2 + 6x = U$, maka $du = (2x + 6)\, dx$

atau $dU = 2(x + 3)\, dx$.

$$\int \frac{(x + 3)\, dx}{(x^2 + 6x)^{1/3}} = \int U^{-1/3} \cdot \tfrac{1}{2}\, dU$$

$$= \frac{3}{4} U^{2/3} + C = \frac{3}{4} (x^2 + 6x)^{2/3} + C.$$

12. $\int \frac{(x^2 + 1)\, dx}{\sqrt[3]{(x^3 + 3x)}} = \frac{1}{3}\int (x^3 + 3x)^{-1/3} (3x^2 + 3)\, dx$

$$= \frac{1}{3}\int (x^3 + 3x)^{-1/3}\, d(x^3 + 3x)$$

$$= \frac{1}{3} \cdot \frac{3}{2} (x^3 + 3x)^{2/3} + C = \tfrac{1}{2} (x^3 + 3x)^{2/3} + C.$$

13. $\int \frac{(\sqrt{a} - \sqrt{x})^2}{\sqrt{x}}\, dx = \int (\sqrt{a} - \sqrt{x})^2 x^{-\frac{1}{2}}\, dx = -2\int (\sqrt{a} - \sqrt{x})^2\, d(\sqrt{a} - \sqrt{x})$

$$= -\frac{2}{3} (\sqrt{a} - \sqrt{x})^3 + C.$$

14. $\int \sqrt{x}\, (\sqrt{a} - \sqrt{x})^2\, dx = \int \sqrt{x}\, (a - 2\sqrt{ax} + x)\, dx$

$$= \int (a\sqrt{x} - 2x\sqrt{a} + x\sqrt{x})\, dx$$

$$= \frac{2}{3} ax^{3/2} - x^2 a + \frac{2}{5} x^{5/2} + C.$$

15. $$\int \frac{x^2 + 2x}{(x+1)^2}\,dx = \int \frac{x^2 + 2x - 1 - 1}{(x+1)^2}\,dx = \int \frac{(x+1)^2 - 1}{(x+1)^2}\,dx$$

$$= \int \left\{ 1 - \frac{1}{(x+1)^2} \right\} dx$$

$$= x + \frac{1}{x+1} + C$$

Soal-soal sesuai dengan rumus 2.5 s/d 2.7

16. $$\int \frac{dx}{x} = \ln x + C.$$

17. $$\int \frac{dx}{2+3x} = \frac{1}{3}\int \frac{3\,dx}{2+3x} = \frac{1}{3}\int \frac{d(2+3x)}{2+3x} = \frac{1}{3}\ln|2+3x| + C.$$

18. $$\int \frac{(2+\ln x)}{x}\,dx = \int (2+\ln x)\,d(\ln x) = \int (2+\ln x)\,d(2+\ln x)$$

$$= \tfrac{1}{2}(2+\ln x)^2 + C.$$

19. $$\int \frac{x\,dx}{x^2-1} = \tfrac{1}{2}\int \frac{2x\,dx}{x^2-1} = \tfrac{1}{2}\int \frac{d(x^2-1)}{x^2-1} = \tfrac{1}{2}\ln|x^2-1| + C.$$

20. $$\int \frac{x^2\,dx}{1-2x^3} = -\frac{1}{6}\int \frac{-6x^2\,dx}{1-2x^3} = -\frac{1}{6}\int \frac{d(1-2x^3)}{1-2x^3}$$

$$= -\frac{1}{6}\ln|1-2x^3| + C.$$

21. $$\int \frac{(2x+3)\,dx}{x+2} = \int \left(2 - \frac{1}{x+2}\right) dx = 2x - \ln|x+2| + C.$$

22. $$\int \frac{(x+4)\,dx}{2x+3} = \int \left(\tfrac{1}{2} + \frac{5}{2(2x+3)}\right) dx = \frac{x}{2} + \frac{5}{4}\ln|2x+3| + C.$$

23. $$\int e^{-x}\,dx = -\int e^{-x}\,d(-x) = -e^{-x} + C.$$

24. $$\int 10^x\,dx = \frac{10^x}{\ln 10} + C.$$

25. $\int \frac{e^{\sqrt{x}}\, dx}{x} = 2\int e^{\sqrt{x}}\, d\sqrt{x} = 2\, e^{\sqrt{x}} + C.$

26. $\int 6\, e^{3x}\, dx = 2\int e^{3x}\, d\,(3x) = 2\, e^{3x} + C.$

27. $\int a^x\, e^x\, dx = \int (ae)^x\, dx = \frac{(ae)^x}{\ln\,(ae)} + C = \frac{a^x\, e^x}{1 + \ln a} + C.$

28. $\int (e^x + 1)^3\, e^x\, dx$, untuk mempermudah penyelesaian, dimisalkan :

$$e^x + 1 = U \longrightarrow dU = e^x\, dx.$$

Maka : $\int (e^x + 1)^3\, e^x\, dx = \int U^3\, dU = \frac{1}{4}\, U^{1/4} + C = \frac{1}{4}\, (e^x + 1)^4 + C.$

29. $\int \frac{e^{2x}\, dx}{e^{2x} + 1}$, misalkan $e^{2x} + 1 = U \longrightarrow dU = 2\, e^{2x}\, dx.$

maka : $\int \frac{e^{2x}\, dx}{e^{2x} + 1} = \frac{1}{2}\int \frac{dU}{U} = \frac{1}{2} \ln U + C. = \frac{1}{2} \ln\, e^{2x} + 1 + C.$

30. $\int \frac{a\, e^x + b}{a\, e^x - b}\, dx = \int \frac{2\, a\, e^x - a\, e^x + b}{a\, e^x - b}\, dx$

$$= 2\int \frac{a\, e^x}{a\, e^x - b}\, dx - \int dx = 2 \ln \left| a\, e^x - b \right| + x + C.$$

31. $\int \frac{dx}{e^x + 1} = \int \frac{e^x\, dx}{1 + e^{-x}} = -\ln \left| 1 + e^{-x} \right| + C = \ln \left| \frac{e^x}{1 + e^x} \right| + C.$

Oleh karena $1 + e^x > 0$ untuk semua harga x, maka hasil integrasi dapat ditulis :

$$\int \frac{dx}{e^x + 1} = \ln \frac{e^x}{1 + e^x} + C.$$

32. $\int \frac{e^{1/x}\, dx}{x^2} = \int e^{1/x} \left(\frac{dx}{x^2} \right) = -\int e^{1/x}\, d\,(1/x) = -\, e^{1/x} + C.$

33. $\int (e^{5x} + a^{5x})\, dx = \frac{1}{5} e^{5x} + \frac{a^{5x}}{5 \ln a} + C = \frac{1}{5} (e^{5x} + \frac{a^{5x}}{\ln a}) + C.$

Soal-Soal sesuai dengan rumus 2.8 s/d 2.17

34. $\int \sin \tfrac{1}{2}x\, dx = 2 \int \sin \tfrac{1}{2}x \,.\, d\tfrac{1}{2}x = -2 \cos \tfrac{1}{2} x + C.$

35. $\int \sin 4x\, dx = \frac{1}{4} \int \sin 4x\, d(4x) = -\frac{1}{4} \cos 4x + C.$

36. $\int \text{Cos}^2 2x \sin 2x\, dx = \tfrac{1}{2} \int \cos^2 2x \sin 2x\, d(2x)$

$= \tfrac{1}{2} \int \cos^2 2x\, (\sin 2x\, d\, 2x)$

$= -\tfrac{1}{2} \int \cos^2 2x\, d(\cos 2x) = -\frac{1}{6} \cos^3 2x + C.$

37. $\int \cos 3ax\, dx = \frac{1}{3a} \int \cos 3ax\, d(3ax) = \frac{1}{3a} \sin 3ax + C.$

38. $\int \text{tg } 2x\, dx = \tfrac{1}{2} \int \frac{\sin 2x}{\cos 2x} d(2x) = -\tfrac{1}{2} \int \frac{d(\cos 2x)}{\cos 2x} = \tfrac{1}{2} \ln \sec 2x + C.$

39. $\int x \,\text{xtg}\, x^2\, dx = \tfrac{1}{2} \int \text{ctg } x^2\, d(x^2) = \tfrac{1}{2} \ln \sin x^2 + C.$

40. $\int (\text{tg } 2x - 1)^2\, dx = \int (\text{tg}^2 2x - 2 \text{ tg } 2x + 1)\, dx$

$= \int \left[(\text{tg}^2 2x + 1) - 2 \frac{\sin 2x}{\cos 2x} \right] dx$

$= \int (\sec^2 2x - 2 \frac{\sin 2x}{\cos 2x})\, dx$

$= \tfrac{1}{2} \text{ tg } 2x + \ln \cos 2x + C.$

41. $\int x^2 \sec^2 x^3\, dx = \frac{1}{3} \int \sec^2 x^3\, dx^3 = \frac{1}{3} \text{ tg } x^3 + C.$

42. $\int \sec 3t \text{ tg } 3t\, dt = \frac{1}{3} \int \sec 3t \,.\, \text{tg } 3t\, d(3t) = \frac{1}{3} \sec 3t + C.$

43. $\int \text{cosec}^2\ 3x\ dx = -\frac{1}{3}\ \text{ctg}\ 3x + C.$

44. $\int (\sec \phi - \text{tg}\ \phi)^2 = \int (\sec^2 \phi - 2 \sec \phi\ \text{tg}\ \phi + \text{tg}^2\ \phi)\ d\phi$

$= \int (2 \sec^2 \phi - 2 \sec \phi\ \text{tg}\ \phi - 1)\ d\phi$

$= 2\ \text{tg}\ \phi - 2 \sec \phi - \phi + C.$

45. $\int \frac{\sin\ x\ dx}{\sqrt{x}} = 2\int \sin \sqrt{x}\ d\sqrt{x} = -\cos s \sqrt{x} + C.$

46. $\int \frac{\sec^2 \phi}{\sqrt{1 + 2\ \text{tg}\ \phi}}\ d\phi = \int \frac{d\ (\text{tg}\ \phi)}{\sqrt{1 + 2\ \text{tg}\ \phi}} = \sqrt{1 + 2\ \text{tg}\ \phi} + C.$

47. $\int \frac{dx}{\text{cosec}\ 2x - \text{ctg}\ 2x} = \int \frac{\sin 2x\ dx}{1 - \cos 2x} = -\frac{1}{2}\int \frac{d\ (\cos 2x)}{1 - \cos 2x}$

$= \frac{1}{2}\ \ln\ (1 - \cos 2x) + C.$

Soal-soal sesuai dengan rumus 2.18 s/d 2.20

48. $\int \frac{dx}{x^2 + 9} = \frac{1}{3}\ \text{arc tg}\ \frac{x}{3} + C.$

49. $\int \frac{d\phi}{\sqrt{25 - \phi^2}} = \text{arc sin}\ \frac{\phi}{5} + C$

50. $\int \frac{dx}{x\sqrt{x^2 - 9}} = \frac{1}{3}\ \text{arc sec}\ \frac{x}{3} + C.$

51. $\int \frac{e^x\ dx}{1 + e^{2x}} = \frac{de^x}{1 + (e^x)^2} = \text{arc tg}\ e^x + C.$

52. $\int \frac{5x\ dx}{\sqrt{1 - x^4}} = \frac{5}{2}\ \frac{dx^2}{\sqrt{1 - (x^2)^2}} = \frac{5}{2}\ \text{arc sin}\ x^2 + C.$

53. $\int \frac{dx}{x^2 - 4x + 13}$. Pandang bentuk $x^2 - 4x + 13$, dapat ditulis menjadi :

$$x^2 - 4x + 4 + 9 = (x - 2)^2 + 9$$

Maka :

$$\int \frac{dx}{x^2 - 4x + 13} = \int \frac{dx}{(x - 2)^2 + 9} = \int \frac{d(x-2)}{(x-2)^2 + 3^2}$$

$$= \frac{1}{3} \text{ arc tg } \frac{x - 2}{3} + C.$$

54. $\int \frac{dx}{x^2 + 2x + 5} = \int \frac{d(x+1)}{(x+1)^2 + 2^2} = \frac{1}{2} \text{ arc tg } \frac{x + 1}{2} + C.$

55. $\int \frac{2\,dx}{\sqrt{2 + x - x^2}} = \int \frac{2\,dx}{\sqrt{2 - (x^2 - x)}} = \int \frac{dx}{\sqrt{\frac{9}{4} - (x - \frac{1}{2})^2}}$

$$= 2 \text{ arc sin } \frac{x - \frac{1}{2}}{3/2} + C = 2 \text{ arc sin } \frac{2x - 1}{3} + C.$$

Soal-soal sesuai dengan rumus 2.21 s/d 2.26

56. $\int \frac{dx}{x^2 - 4} = \frac{1}{4} \ln \frac{x - 2}{x + 2} + C.$

57. $\int \frac{dx}{9x^2 - 4} = \frac{1}{3} \int \frac{d(3x)}{9x^2 - 4} = \frac{1}{12} \ln \left| \frac{3x - 2}{3x + 2} \right| + C.$

58. $\int \frac{dt}{4 - 9t^2} = \frac{1}{3} \int \frac{d(3t)}{2^2 - (3t)^2} = \frac{1}{12} \ln \left| \frac{2 + 3t}{2 - 3t} \right| + C.$

59. $\int \frac{\cos \phi \, d\phi}{4 - \sin^2 \phi} = \int \frac{d \sin \phi}{4 - \sin^2 \phi} = \frac{1}{4} \ln \left| \frac{2 + \sin \phi}{2 - \sin \phi} \right| + C.$

60. $\int \frac{dx}{\sqrt{1 + x + x^2}} = \int \frac{dx}{\sqrt{(x + \frac{1}{2})^2 + \frac{3}{4}}} = \ln \left| x + \frac{1}{2} + \sqrt{x^2 + x + 1} \right| + C.$

61. $\int \frac{dx}{\sqrt{2ax + x^2}} = \int \frac{dx}{\sqrt{(x + a)^2 - a^2}} = \ln\ x + a + \sqrt{2ax + x^2} + C.$

62. $\int \sqrt{1 - 4x^2}\, dx = \frac{1}{2} \int \sqrt{1 - (2x)^2}\, d(2x).$

$= \frac{x}{2}\sqrt{1 - 4x^2} + \frac{1}{4} \arcsin 2x + C.$

63. $\int \sqrt{5 - 3x}\, dx = \frac{x}{2}\sqrt{5 - 3x^2} + \frac{5}{2\sqrt{3}} \arcsin \left(x\sqrt{\frac{3}{5}}\right) + C$

64. $\int \sqrt{9x^2 + 1}\, dx = \frac{x}{2}\sqrt{9x^2 + 1} + \frac{1}{6} \ln 3x + \sqrt{9x^2 + 1} + C.$

65. $\int \sqrt{4x^2 + 9}\, dx = \frac{x}{2}\sqrt{4x^2 + 9} + \frac{9}{4} \ln 2x + \sqrt{4x^2 + 9} + C.$

66. $\int \sqrt{x^2 - 8}\, dx = \frac{x}{2}\sqrt{x^2 - 8} - 4 \ln x + \sqrt{x^2 - 8} + C.$

Contoh soal-soal campuran :

67. $\int \frac{(1 + 2x)}{1 + x^2}\, dx = \int \frac{1}{1 + x^2}\, dx + \frac{2x\, dx}{1 + x^2}$

$= \text{arc tg } x + \ln (1 + x^2) + C.$

68. $\int \frac{2x + 1}{\sqrt{x^2 - 1}}\, dx = \int \frac{2x\ dx}{\sqrt{x^2 - 1}} + \frac{1}{\sqrt{x^2 - 1}}\, dx$

$= 2\sqrt{x^2 - 1} + \ln (x + x^2 - 1) + C.$

69. $\int \frac{(x - 1)}{\sqrt{1 - x^2}}\, dx = \int \frac{x\ dx}{\sqrt{1 - x^2}} - \int \frac{dx}{\sqrt{1 - x^2}}$

$= -\sqrt{1 - x^2} - \arcsin x + C.$

70. $\int \frac{(x+3)}{\sqrt{x^2+4}}\,dx = \int \frac{x\,dx}{\sqrt{x^2+4}} + \int \frac{3\,dx}{\sqrt{x^2+4}}$

$= \sqrt{x^2+4} + 3 \ln\left(x + \sqrt{x^2+4}\right) + C.$

71. $\int \sqrt{10 - 4x + x^2}\,dx = \int \sqrt{10 - 4 + (x-2)^2}\,dx.$

$= \int \sqrt{6 + (x-2)^2}\,dx.$

$= \frac{x-2}{2}\sqrt{10 - 4x + x^2}.$

$+ 3 \ln\left(x - 2 + \sqrt{10 - 4x + x^2}\right) + C.$

72. $\int \sqrt{3 - 2x - x^2}\,dx = \int \sqrt{4 - (x+1)^2}\,dx$

$= \frac{x+1}{2}\sqrt{3 - 2x - x^2} + 2 \arcsin\frac{x+1}{2} + C.$

73. $\int \frac{(1-x)\,dx}{4x^2 - 4x - 3} = -\frac{1}{8}\int \frac{8x - 4 - 4}{4x^2 - 4x - 3}\,dx$

$= -\frac{1}{8}\int \frac{(8x-4)\,dx}{4x^2 - 4x - 3} + \frac{1}{2}\frac{dx}{(2x-1)^2 - 4}$

$= -\frac{1}{8}\ln(4x^2 - 4x - 3) + \frac{1}{16}\ln\left|\frac{2x-3}{2x+1}\right| + C$

74. $\int \frac{(3x-2)\,dx}{1 - 6x - 9x^2} = \frac{1}{6}\int \frac{(18x - 12)\,dx}{1 - 6x - 9x^2} = -\frac{1}{6}\int \frac{(-18x - 6) + 18}{1 - 6x - 9x^2}\,dx$

$= -\frac{1}{6}\int \frac{(18x - 6)\,dx}{1 - 6x - 9x^2} - 3\int \frac{dx}{2 - (3x+1)^2}$

$= -\frac{1}{6}\ln\left|1 - 4x - 9x^2\right| + \frac{1}{2\sqrt{2}}\ln\left|\frac{3x + 1 - \sqrt{2}}{3x + 1 + \sqrt{2}}\right| + C$

$$75.\int \frac{x+2}{\sqrt{4x-x^2}}\,dx = -\tfrac{1}{2}\int \frac{-2x-4}{\sqrt{4x-x^2}}\,dx = -\tfrac{1}{2}\int \frac{(-2x+4)-8}{\sqrt{4x-x^2}}\,dx$$

$$= -\tfrac{1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx + 4\int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= -\sqrt{4x-x^2} + 4 \operatorname{arc\,sin} \frac{x-2}{2} + C.$$

$$76.\int \frac{(2x+3)\,dx}{9x^2-12x+8} = \frac{1}{9}\int \frac{(18x+27)\,dx}{9x^2-12x+8}$$

$$= \frac{1}{9}\int \frac{(18x-12)+39}{9x^2-12x+8}\,dx$$

$$= \frac{1}{9}\int \frac{18x-12}{9x^2-12x+8}\,dx + \frac{13}{3}\int \frac{dx}{(3x-2)^2+4}$$

$$= \frac{1}{9}\ln(9x^2-12x+8) + \frac{13}{18}\operatorname{arc\,tg}\frac{3x-2}{2} + C$$

$$77.\int \frac{x+2}{\sqrt{5-4x-x^2}}\,dx = -\tfrac{1}{2}\int \frac{(-2x-6)\,dx}{\sqrt{5-4x-x^2}} = -\tfrac{1}{2}\int \frac{(-2x-4)-2}{\sqrt{5-4x-x^2}}\,dx$$

$$= -\tfrac{1}{2}\int \frac{(-2x-4)\,dx}{\sqrt{5-4x-x^2}} + \int \frac{dx}{\sqrt{9-(x+2)^2}}$$

$$= -\sqrt{5-4x-x^2} + \operatorname{arc\,sin}\frac{x+2}{3} + C.$$

$$78.\int \frac{2x^4-x^2}{2x^2+1} + dx = \int \left(x^2 - 1 + \frac{1}{2x^2+1}\right) dx = \frac{1}{3}x^2 - x$$

$$+\int \frac{1}{2x^2+1}\,dx$$

$$= \frac{1}{3}x^3 - x + \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg} x\sqrt{2} + C.$$

79. $\int \frac{dx}{x\sqrt{4 - 9\ln^2 x}} = \int \frac{d(\ln x)}{\sqrt{4 - 9\ln^2 x}} = \frac{1}{3} \text{ ar sin } (\frac{3}{2} \ln x) + C$

80. $\int \frac{\sin 8x}{9 + \sin^4 4} dx = 2\int \frac{\sin 4x \cos 4x}{9 + \sin^4 4x} dx = \frac{1}{4}\int \frac{2 \sin 4x \, d(\sin 4x)}{9 + \sin^4 4x}$

$= \frac{1}{4}\int \frac{d(\sin^2 4x)}{9 + \sin^4 4x} = \frac{1}{12} \text{ arc tg } \frac{\sin^2 4x}{3} + C.$

81. $\int \frac{(2x - 1)\,dx}{4x^2 - 11} = \frac{1}{4}\int \frac{8x - 12}{4x^2 - 11} dx$

$= \frac{1}{4}\int \frac{8x\,dx}{4x^2 - 11} - \frac{12}{4}\int \frac{dx}{4x^2 - 11}$

$= \frac{1}{4}\ln (4x^2 - 11) - \frac{3\sqrt{11}}{\sqrt{44}} \ln \left(\frac{2x - \sqrt{11}}{2x + \sqrt{11}}\right) + C.$

82. $\int \frac{(x + 2)\,dx}{\sqrt{x^2 + 2x - 3}} = \frac{1}{2}\int \frac{(2x + 4)\,dx}{\sqrt{x^2 + 2x - 3}} = \frac{1}{2}\int \frac{(2x + 2) + 2}{\sqrt{x^2 + 2x - 3}} dx$

$= \frac{1}{2}\int \frac{(2x + 2)\,dx}{\sqrt{x^2 + 2x - 3}} + \int \frac{dx}{\sqrt{(x + 1)^2 - 4}}$

$= \sqrt{x^2 + 2x - 3} + \ln (x + 1 + \sqrt{x^2 + 2x - 3}) + C.$

83. $\int \frac{(2 - x)\,dx}{4x^2 + 4x - 3} = -\frac{1}{8}\int \frac{(8x - 16)\,dx}{4x^2 + 4x - 3} = -\frac{1}{8}\int \frac{(8x + 4) - 20}{4x^2 + 4x - 3} dx$

$= -\frac{1}{8}\int \frac{8x + 4}{4x^2 + 4x - 3} dx + \frac{20}{8}\int \frac{dx}{(2x + 1)^2 - 4}$

$= -\frac{1}{8}\ln (4x^2 + 4x - 3) + \frac{5}{16}\ln \left|\frac{2x - 1}{2x + 3}\right| + C.$

84. $\int \sqrt{3x^2 + 5}\, dx = \frac{1}{\sqrt{3}}\int \sqrt{3x^2 + 5}\, d(x\sqrt{3})$

$= \frac{1}{\sqrt{3}}\left| \frac{3}{2} x \sqrt{3x^2 + 5} + \frac{5}{2} \ln (x\sqrt{3} + \sqrt{3x^2 + 5}) \right| + C.$

$$= \tfrac{1}{2}\, x\sqrt{3x^2 + 5} + \frac{5\sqrt{3}}{6} \ln\left(x\sqrt{3} + \sqrt{3x^2 + 5}\right) + C.$$

85. $\displaystyle\int \frac{1 + \cos x}{x + \sin x}\, dx = \int \frac{d(x + \sin x)}{x + \sin x} = \ln(x + \sin x) + C.$

Soal - Soal :

1. $\displaystyle\int x^4\, dx$
2. $\displaystyle\int \frac{dx}{x^2}$
3. $\displaystyle\int x^{2/3}\, dx$
4. $\displaystyle\int \frac{dx}{\sqrt{x}}$
5. $\displaystyle\int \frac{1}{\sqrt[3]{x}}\, dx$
6. $\displaystyle\int \sqrt{ax}\, dx$
7. $\displaystyle\int \sqrt[3]{3t}\, dt$
8. $\displaystyle\int (x^{3/2} - 2x^2 + 5\sqrt{x} - 3)\, dx.$
9. $\displaystyle\int \frac{4x^2 - 2\sqrt{x}}{x}\, dx$
10. $\displaystyle\int \sqrt{x}\,(3x - 2)\, dx$
11. $\displaystyle\int (1 - t)^2\, dt$
12. $\displaystyle\int t\,(t^2 + 1)^2\, dt.$
13. $\displaystyle\int \sqrt{x - 6}\, dx$
14. $\displaystyle\int x\,(x^2 - 9)^{3/2}\, dx$
15. $\displaystyle\int \frac{2x + 1}{x^2 + x + 1}\, dx$
16. $\displaystyle\int \frac{x^2 + 4}{x}\, dx$
17. $\displaystyle\int \frac{x\, dx}{x^2 + 16}$
18. $\displaystyle\int \frac{x^2 - 6x + 5}{x}\, dx$
19. $\displaystyle\int \frac{dy}{\sqrt{2 - y}}$
20. $\displaystyle\int \frac{y\, dy}{(2y^2 + 50)^2}$
21. $\displaystyle\int \frac{4x^2\, dx}{\sqrt{x^3 + 8}}$
22. $\displaystyle\int \frac{(x^2 + 1)\, dx}{\sqrt[3]{x^3 + 3x}}$

23. $\int x^{n-1} \sqrt{a + bx^n}\, dx$

24. $\int \frac{x^2 + 4}{x}\, dx$

25. $\int \frac{(x - \ln x)}{x}\, dx$

26. $\int \frac{x}{x^2 + 4}\, dx$

27. $\int \frac{1}{x^2 - 4x + 4}\, dx$

28. $\int \frac{\sin 2x}{\cos^4 2x}\, dx$

29. $\int \frac{2x + 3}{x^2 + 3x}\, dx$

30. $\int \left(\frac{\sec x}{1 + \operatorname{tg} x}\right)^2 dx$

31. $\int \frac{\sqrt{a^{2/3} - x^{2/3}}}{x^{1/3}}\, dx$

32. $\int \frac{\sqrt{\ln x}}{x}\, dx$

33. $\int \frac{dx}{x \ln x}$

34. $\int \frac{(1 + \ln x)^{3/2}}{x}\, dx$

35. $\int \frac{\ln \ln x}{x \ln x}\, dx$

36. $\int e^x (1 + e^x)^5\, dx$

37. $\int \frac{e^{3x}}{1 + e^{3x}}\, dx$

38. $\int \frac{\sqrt{1 + e^{-x}}}{e^x}\, dx$

39. $\int \frac{x^2 + 4}{x - 1}\, dx$

40. $\int \frac{x^3 + 3x}{x^2 + 1}\, dx$

41. $\int \frac{(e^t + 2)\, dt}{c^t + 2t}$

42. $\int \frac{\sec 2x \operatorname{tg} 2\, dx}{3 \sec 2x - 2}$

43. $\int e^{\cos 2x} \sin 2x\, dx$

44. $\int x\, 2^{x^2}\, dx$

45. $\int \frac{dx}{4^{2x}}$

46. $\int (e^{x^2} + 16)\, x\, e^{x^2}\, dx$

47. $\int \frac{10^x - 1}{10^x + 1}\, dx$

48. $\int 10^{\operatorname{ctg} 3x} \operatorname{cosec}^2 3x\, dx$

49. $\int \frac{(e^{4x} - 2e^{3x} + 5e^x - 2)\, dx}{e^x + 1}$

50. $\int \frac{e^x\, dx}{1 + e^{\frac{1}{2}x}}$

51. $\int \cos nx\, dx$

52. $\int x \cos x^2 \sin^2 x^2\, dx$

53. $\int \frac{1}{x^2} \sin \frac{\pi}{x}\, dx$

54. $\int \operatorname{cosec} \left(5x + \frac{\pi}{7}\right) dx$

55. $\int \frac{\cos z + \sin z}{\sin^2 z}\, dz$

56. $\int \frac{dx}{\cos x \operatorname{ctg} x}$

57. $\int \frac{\sin y + \cos y}{\sin y}\, dy$

58. $\int \frac{(1 + \operatorname{tg} 2x)^2}{\cos^2 2x}\, dx$

59. $\int \frac{dx}{\sin 3x \operatorname{tg} 3x}\, dx$

60. $\int \frac{\cos^3 x}{1 + \sin x}\, dx$

61. $\int \frac{\sin^3 x}{1 - \cos x}\, dx$

62. $\int \frac{\operatorname{tg}^2 x}{\sec x - 1}\, dx$

63. $\int \frac{\operatorname{ctg}^2 x}{1 - \operatorname{cosec} x}\, dx$

64. $\int \frac{dx}{1 + \cos x}$

65. $\int \frac{dx}{1 + \operatorname{Cos} x}$

66. $\int \frac{dx}{1 + \sec ax}$

67. $\int \frac{e^{2x} - 1}{e^{2x} + 3}\, dx$

68. $\int \left(e^x + \frac{1}{e^x}\right)^2 dx$

69. $\int \frac{e^x - 1}{e^x + 1}\, dx$

70. $\int \frac{\sec^5 x}{\operatorname{cosec} x}\, dx$

71. $\int (e^x - x^e)\, dx$

72. $\int \frac{dx}{\sqrt{5 - x^2}}$

73. $\int \frac{dx}{5x + x^2}$

74. $\int \frac{dx}{x\sqrt{x^2 - 5}}$

75. $\int \frac{e^{2x}}{1 + e^{4x}}\, dx$

76. $\int \frac{dx}{\sqrt{4 - 9x^2}}$

77. $\int \frac{dx}{9x^2 + 4}$

78. $\int \frac{\sin 18x}{9 + \sin^4 9x}\, dx$

79. $\int \frac{\sec^2 x}{\sqrt{1 - 4\,\mathrm{tg}^2 x}}\, dx$

80. $\int \frac{dx}{x\sqrt{4 - 31\,\ln^2 2x}}$

81. $\int \frac{2x^4 - 3x^2}{2x^2 + 1}\, dx$

82. $\int \frac{\cos 2x\, dx}{\sin^2 2x + 1}$

83. $\int \frac{dx}{\sqrt{3x - x^2 - 2}}$

84. $\int \frac{dx}{2x - x^2 - 10}$

85. $\int \frac{dx}{x^2 - 8x + 25}$

86. $\int \frac{dx}{\sqrt{15 - 2x - x^2}}$

87. $\int \frac{dx}{1 + x + x^2}$

88. $\int \frac{(2x - 3)\, dx}{x^2 + 6x + 13}$

89. $\int \frac{(x - 1)\, dx}{3x^2 - 4x + 3}$

90. $\int \frac{x\, dx}{\sqrt{27 + 6x - x^2}}$

91. $\int \frac{(5 - 4x)\, dx}{\sqrt{12x - 4x^2 - 8}}$

92. $\int \frac{(8x - 3)\, dx}{\sqrt{12x - 4x^2 - 5}}$

93. $\int \frac{x\, dx}{\sqrt{27 + 6x - x^2}}$

94. $\int \frac{4x + 3}{x^2 + 1}\,dx$

95. $\int \frac{dx}{x^2 - 4x + 3}$

96. $\int \frac{dx}{x^2 - 6x + 5}$

97. $\int \frac{dx}{x^2 + 2x}$

98. $\int \frac{dx}{4x - x^2}$

99. $\int \frac{dy}{y^2 + 3y + 1}$

100. $\int \frac{dx}{\sqrt{2bx + b^2x^2}}$

101. $\int \frac{5\,dx}{\sqrt{x^2 + 2x + 5}}$

102. $\int \frac{dx}{\sqrt{x^2 + 4x + 3}}$

103. $\int \frac{dy}{\sqrt{3y - 2y^2}}$

104. $\int \frac{dx}{2 + 2x + x^2}$

105. $\int \frac{dt}{t^2 - 2t - 3}$

106. $\int \frac{4\,dx}{\sqrt{x^2 - 4x + 13}}$

107. $\int \frac{dU}{\sqrt{8u^2 - 8u + 15}}$

108. $\int \frac{x\,dx}{x^4 - x^2 - 1}$

109. $\int \frac{dx}{3x^2 + 4x + 1}$

110. $\int \frac{x^2\,dx}{9x^6 - 3x^3 - 1}$

111. $\int \frac{dt}{15 + 4t - t^2}$

112. $\int \frac{2x + 3}{\sqrt{x^2 - 2}}\,dx$

113. $\int \frac{x + 3}{\sqrt{x^2 + 16}}\,dx$

114. $\int \frac{(2x - 5)\,dx}{3x^2 - 2}$

115. $\int \frac{(5x - 1)\,dx}{\sqrt{3x^2 - 9}}$

116. $\int \frac{(x + 1)\,dx}{6x - x^2}$

117. $\int \frac{(x + 3)\,dx}{\sqrt{x^2 + 2x}}$

118. $\int \frac{(3x + 2)\ dx}{\sqrt{19 - 5x + x^2}}$

119. $\int \frac{(3x - 2)\ dx}{\sqrt{4x^2 - 4x + 5}}$

120. $\int \frac{(4x + 5)\ dx}{\sqrt{3x - x^2}}$

121. $\int \frac{(4x + 3)\ dx}{x^2 + 1}$

122. $\int \frac{(3x - 4)\ dx}{4 - 3x^2}$

123. $\int \frac{(3 - x)\ dx}{4 - 3x^2}$

124. $\int \frac{(2x + 3)\ dx}{\sqrt{2 - 3x^2}}$

125. $\int \frac{(4x - 1)\ dx}{\sqrt{3 + 5x^2}}$

126. $\int \frac{(3x - 5)\ dx}{x^2 + 4x}$

127. $\int \frac{(4x + 5)\ dx}{\sqrt{3x - x^2}}$

128. $\int \frac{(x + 2)\ dx}{x^2 - 6x + 5}$

129. $\int \frac{(3 - 4x)\ dx}{\sqrt{3x - x^2 - 2}}$

130. $\int \frac{(5x + 2)\ dx}{\sqrt{x^2 + 2x + 5}}$

131. $\int \frac{(1 - x)\ dx}{\sqrt{x^2 + 4x + 3}}$

132. $\int \sqrt{16 - 9x^2}\ dx$

133. $\int \sqrt{4 + 25x^2}\ dx$

134. $\int \sqrt{x^2 + 1/4\ x^2}\ dx$

135. $\int \sqrt{5 - 2x + x^2}\ dx$

136. $\int \sqrt{2x - x^2}\ dx$

137. $\int \sqrt{9x^2 - 1}\ dx$

138. $\int \sqrt{8 - 3x^2}\ dx$

139. $\int \sqrt{5 + 2x^2}\ dx$

140. $\int \sqrt{5 - 4x - x^2}\ dx$

141. $\int \sqrt{5 + 2x + x^2}\ dx$

142. $\int \sqrt{x^2 - 8x + 7}\ dx$

143. $\int \sqrt{4 - 2x - x^2}\ dx$

144. $\int \sqrt{36 - 9x^2}\ dx$

145. $\int \sqrt{x^2 - 16}\ dx$

146. $\int \sqrt{4x^2 + 9x}\ dx$

147. $\int \sqrt{x^2 - 2x - 3}\ dx$

148. $\int \frac{(8 - 3x)\ dx}{x^2 + x + 1}$

149. $\int \frac{(x + 4)\ dx}{\sqrt{x^2 + x + 1}}$

150. $\int \frac{(2x + 7)\ dx}{2x^2 + 2x - 1}$

151. $\int \frac{(3x + 8)\ dx}{9x^2 - 3x - 1}$

152. $\int \frac{(6 - x)\ dx}{\sqrt{4x^2 - 12x + 7}}$

153. $\int \frac{x^2}{\sqrt{9 + x^2}}\ dx$

154. $\int \sqrt{12 + 4x - x^2}\ dx$

155. $\int \sqrt{x^2 + 4x}\ dx$

156. $\int \sqrt{x^2 - 8x}\ dx$

157. $\int \sqrt{6x - x^2}\ dx$

158. $\int \sqrt{x^2 - 2x + 2}\ dx$

159. $\int \sqrt{x^2 + x}\ dx$

160. $\int \sqrt{2 + x^2}\ dx$

161. $\int \sqrt{x^2 - 2x + 8}\ dx$

162. $\int \sqrt{10 - 4x + 4x^2}\ dx$

163. $\int \sqrt{3 - 2x - x^2}\ dx$

164. $\int \sqrt{x^2 - 6x - 7}\ dx$

165. $\int (x^2 + x + 1)^{3/2}\ dx$

166. $\int \sqrt{x^2 - 2x + 2}\ dx$

167. $\int \sqrt{e^x + 1}\ dx$

168. $\int \sqrt{3 + 2x - x^2}\ dx$

169. $\int \sqrt{x^2 - 6x + 11}\ dx$

3. INTEGRASI PARSIIL (INTEGRATION BY PARTS)

Suatu bentuk integral yang sering timbul, ialah suatu integral yang integrannya merupakan hasil ganda dari suatu fungsi x, dengan differensial dari fungsi x yang lain.

Andaikan U dan V adalah fungsi dari x, maka dicari hasil dari bentuk :

$$\int U.dV$$

Dalam hitung differensial telah diketahui, bahwa :

$$d(U.V) = U\,dV + V\,dU$$

atau

$$U\,dV = d\,(U.V) - V\,dU$$

Maka :

$$\int U\,dV = U.V - \int V\,dU$$

Agar kita dapat menggunakan rumus ini, bentuk integran dari integral yang diketahui, harus dibuat menjadi dua bagian : satu bagian sesuai dengan U dan bagian yang lain bersama-sama dengan dx sesuai dengan dV.

Untuk jelasnya, kita ambil beberapa contoh soal :

1. Tentukan $\int x \cos 2x\,dx$.

   Penyelesaian :

   Disini ada beberapa pilihan atau kemungkinan :

   a). $U = x \cos 2x$, $dV = dx$; b). $U = \cos 2x$, $dV = x\,dx$

   c). $U = x$, $dV = \cos 2x\,dx$.

   a). Untuk $U = x \cos 2x$, $dU = (\cos 2x - 2x \sin 2x)\,dx$

   $dV = dx$, $V = x$

   Maka :

   $$\int x \cos 2x\,dx = x.x \cos 2x - \int x\,(\cos 2x - 2x \sin 2x)dx.$$

   Ternyata hasil integralnya tidak lebih sederhana dari pada integral yang semula dan bentuk ini tidak dipilih.

b) Untuk $U = \cos 2x$, $dU = -2 \sin 2x\, dx$

$dV = x\, dx = \frac{1}{2} dx^2$, $V = \frac{1}{2} x^2$

Maka :

$$\int x \cos 2x\, dx = \frac{1}{2} x^2 \cos 2x + \int x^2 \sin 2x\, dx.$$

Ternyata hasil integralnya juga tidak lebih sederhana dari pada integral yang semula, dan bentuk ini tidak diambil (dipilih).

c) Untuk $U = x$, $dU = dx$

$dV = \cos 2x\, dx$, $V = \frac{1}{2} \sin 2x\, dx$

M a k a :

$$\int x \cos 2x\, dx = \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x\, dx$$
$$= \frac{1}{2} x \sin 2x + \frac{1}{4} \cos 2x + C.$$

2. Tentukan : $\int x e^x\, dx.$

Penyelesaian :

D i a m b i l : $U = x$, $dU = dx$

$dV = e^x\, dx = de^x$, $V = e^x$

M a k a :

$$\int x e^x\, dx = x e^x - \int e^x\, dx$$
$$= x e^x - e^x + C.$$

3. Tentukan : $\int x^2 \ln x\, dx.$

Penyelesaian :

Diambil $U = \ln x$, $dU = \frac{1}{x} dx$

$dV = x^2 dx = d\left(\frac{1}{3} x^3\right)$, $V = \frac{1}{3} x^3$

M a k a :

$$\int x^2 \ln x\, dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^3 \cdot \frac{1}{x} dx.$$
$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.$$

4. Tentukan : $\int \text{arc sin } x\, dx$

Penyelesaian :

D i a m b i l : $U = \arcsin x$, $dU = \frac{1}{\sqrt{1 - x^2}}\, dx$

$dV = dx$, $V = x$.

M a k a :

$$\int \arcsin x\, dx = x \arcsin x - \int x \cdot \frac{1}{\sqrt{1 - x^2}}\, dx$$

$$= x \arcsin x + \sqrt{1 - x^2} + C.$$

5. Tentukan : $\int \operatorname{arc\,tg} x\, dx$.

Penyelesaian :

D i a m b i l : $U = \operatorname{arc\,tg} x$, $dU = \frac{1}{1 + x^2}\, dx$

$dV = dx$, $V = x$

M a k a :

$$\int \operatorname{arc\,tg} x\, dx = x \operatorname{arc\,tg} x - \int x \cdot \frac{1}{1 + x^2}\, dx$$

$$= x \operatorname{arc\,tg} x - \tfrac{1}{2} \ln (1 + x^2) + C.$$

6. Tentukan : $\int \sec^3 x\, dx$

Penyelesaian :

D i a m b i l : $U = \sec x$, $dU = \sec x \operatorname{tg} x\, dx$

$dV = \sec^2 x\, dx = d(\operatorname{tg} x)$, $V = \operatorname{tg} x$

M a k a :

$$\int \sec^3 x\, dx = \sec x \operatorname{tg} x - \int \sec x \operatorname{tg}^2 x\, dx$$

$$= \sec x \operatorname{tg} x - \int \sec x (\sec^2 x - 1)\, dx$$

$$= \sec x \operatorname{tg} x - \int \sec^3 dx + \int \sec x\, dx$$

Suku kedua bagian sebelah kanan, dibawa kekiri, maka terdapat :

$$2 \int \sec^3 x\, dx = \sec x \operatorname{tg} x + \int \sec x\, dx$$

$$\int \sec^3 x\, dx = \tfrac{1}{2} \left[\sec x \operatorname{tg} x + \ln (\sec x + \operatorname{tg} x) \right] + C.$$

7. Tentukan : $\int x^2 \sin x \, dx$

Penyelesaian :

$$\int x^2 \sin x \, dx = -\int x^2 \cos x = -x^2 \cos x + 2 \int x \cos x \, dx$$

$$= -x^2 \cos x + 2 \int x \, d \sin x$$

$$= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos + C.$$

8. Tentukan : $\int e^{2x} \sin x \, dx$

Penyelesaian : Bentuk integral diatas ini dapat diselesaikan dengan menggunakan dua cara.

a) Bila diambil $U = e^{2x}$, $dU = 2 e^{2x} dx$.

$dV = \sin x \, dx = - d \cos x$, $V = - \cos x$.

M a k a :

$$\int e^{2x} \sin x \, dx = - e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx$$

Kemudian diambil lagi : $U = e^{2x}$, $V = \sin x$

$$\int e^{2x} \sin x \, dx = - e^{2x} \cos x + 2 e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx.$$

Suku ketiga dari bagian kanan dibawa kekiri, maka terdapat :

$$\int 5 \, e^{2x} \sin x \, dx = e^{2x} (2 \sin x - \cos x)$$

J a d i :

$$\int e^{2x} \sin x \, dx = \frac{1}{5} e^{2x} (2 \sin x - \cos x) + C.$$

b) Bila diambil $U = \sin x$, $dU = \cos x \, dx$

$dV = e^{2x} dx$, $V = \frac{1}{2} e^{2x}$

M a k a :

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{2} \int e^{2x} \cos x \, dx$$

Kemudian diambil lagi $U = \cos x$, $dU = - \sin x \, dx$

$dV = e^{2x} dx$, $V = \frac{1}{2} e^{2x}$

M a k a :

$$\int e^{2x} \sin x \, dx = \frac{1}{2} e^{2x} \sin x - \frac{1}{4} e^{2x} \cos x$$

$$- \frac{1}{4}\int e^{2x} \sin x \, dx$$

Suku ketiga dari bagian kanan dibawa kekiri, maka terdapat :

$$\frac{5}{4}\int e^{2x} \sin x \, dx = \frac{1}{4} e^{2x} (2 \sin x - \cos x)$$

J a d i :

$$\int e^{2x} \sin x \, dx = - e^{2x} (2 \sin x - \cos x) + C.$$

9. Tentukan :

$$\int \frac{x}{\sqrt{1 - x^2}} \ln \frac{1 + x}{1 - x} \, dx.$$

Penyelesaian :

D i a m b i l : $U = \ln \frac{1+x}{1-x}$, $dU = \frac{1 - x}{1 + x} \cdot \frac{(1 - x).1 - (1 + x)(-1)\, dx}{(1 - x)^2}$

$$= \frac{2 \, dx}{1 - x^2}$$

$$dV = \frac{x}{\sqrt{1 - x^2}} \, dx = - d (\sqrt{1 - x^2}),$$

$$V = -\sqrt{1 - x^2}$$

M a k a :

$$\int \frac{x}{\sqrt{1 - x^2}} \ln \frac{1 + x}{1 - x} \, dx = -\sqrt{1 - x^2} \ln \frac{1 + x}{1 - x} + 2 \int \frac{\sqrt{1 - x^2}}{1 - x^2} \, dx$$

$$\int \frac{x}{\sqrt{1 - x^2}} \ln \frac{1 + x}{1 - x} \, dx = -\sqrt{1 - x^2} \ln \frac{1 + x}{1 - x} + 2 \int \frac{dx}{\sqrt{1 - x^2}}$$

$$= - \sqrt{1 - x^2} \ln \frac{1 + x}{1 - x} + 2 \arcsin x + C.$$

10. Tentukan : $\int \frac{x \arcsin x}{\sqrt{1 - x^2}} \, dx$

Penyelesaian :

D i a m b i l : $U = \arcsin x$, $dU = \frac{1}{\sqrt{1 - x^2}} dx$

$$dV = \frac{x}{\sqrt{1 - x^2}} dx = - d\sqrt{1 - x^2}, \quad V = -\sqrt{1 - x^2}$$

M a k a :

$$\int \frac{x \arcsin x \, dx}{\sqrt{1 - x^2}} = -\sqrt{1 - x^2} \arcsin x + \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}} dx$$

$$= -\sqrt{1 - x^2} \arcsin x + \int dx$$

$$= -\sqrt{1 - x^2} \arcsin x + x + C.$$

11. Tentukan : $\int \sec^5 x \, dx$

Penyelesaian :

D i a m b i l : $U = \sec^3 x$, $dU = 3 \sec^3 x \operatorname{tg} x \, dx$

$dV = \sec^2 x \, dx = d \operatorname{tg} x$, $V = \operatorname{tg} x$.

M a k a : $\int \sec^5 x \, dx = \sec^3 x \operatorname{tg} x - 3 \int \sec^3 x \operatorname{tg}^2 x \, dx$

Sedang : $\operatorname{tg}^2 x = \sec^2 x - 1$,

$$\int \sec^5 x \, dx = \sec^3 x \operatorname{tg} x - 3 \int \sec^3 x (\sec^2 x - 1) \, dx$$
$$= \sec^3 x \operatorname{tg} x - 3 \int \sec^5 dx + 3 \int \sec^3 x \, dx$$

Suku kedua dari bagian kanan dibawa kekiri, sedang :

$$\int \sec^3 x \, dx = \tfrac{1}{2} \left[\sec x \operatorname{tg} x + \ln (\sec x + \operatorname{tg} x) \right] + C.$$

(lihat contoh soal no. 6), maka :

$$4 \int \sec^5 x \, dx = \sec^3 x \operatorname{tg} x + \frac{3}{2} \left[\sec x \operatorname{tg} x + \ln (\sec x + \operatorname{tg} x) \right] + C.$$

J a d i :

$$\int \sec^5 x\,dx = \frac{1}{8}\left\{2\sec^3 x \operatorname{tg} x + 3\left[\sec x \operatorname{tg} x + \ln(\sec x \operatorname{tg} x)\right]\right\} + C.$$

170. $\int \frac{\ln x\,dx}{(x+1)\sqrt{x+1}}$

171. $\int x \sec^2 x\,dx$

172. $\int x^3\sqrt{1+x^2}\,dx$

173. $\int \frac{\ln^2 x\,dx}{x}$

174. $\int x \operatorname{arc\,sec} x\,dx$

175. $\int e^{3x} \sin 2x\,dx$

176. $\int \operatorname{cosec}^3 x\,dx$

177. $\int \ln^2 x\,dx$

178. $\int x\sqrt{x+1}\,dx$

179. $\int x^3 e^{-x^2}\,dx$

180. $\int x^2 e^{3x}\,dx$

181. $\int \frac{\ln(x+1)\,dx}{\sqrt{x+1}}$

182. $\int x^2 \operatorname{arc\,sin} x\,dx$

183. $\int \frac{\ln x}{(x+1)^2}\,dx$

184. $\int x^2 e^{-x}\,dx$

185. $\int \operatorname{arc\,tg}\sqrt{x}\,dx$

186. $\int \operatorname{arc\,cosec} x/2\,dx$

187. $\int \operatorname{arc\,cos} 2x\,dx$

188. $\int x^n \ln x\,dx$

189. $\int x\,a^x\,dx$

190. $\int x \cos^2 2x\,dx$

191. $\int e^{-t/4} \sin(\pi^{t/4})\,dt$

192. $\int \sqrt{1+3\cos^2 x} \sin 2x\,dx$

193. $\int (x^2 - 2x + 5)\,e^{-x}\,dx$

194. $\int (x^2 + 5x + 6) \cos 2x\,dx$

195. $\int \sin(\ln x)\,dx$

196. $\int x \ln \frac{1-x}{1+x}\,dx$

197. $\int x^2\sqrt{1-x}\,dx$

198. $\int (\operatorname{arc\,sin} x)^2\,dx$

199. $\int y^2 \sin ny\,dy$

200. $\int x \sin^2 3x\,dx$

201. $\int x^3 \operatorname{arc\,sin} x\,dx$

202. $\int e^{3x} \cos x/3\,dx$

203. $\int (2^x + x^2)^2 \, dx$

204. $\int \frac{\operatorname{arc\,tg} \sqrt{x}}{x} \, dx$

205. $\int x^2 \cos x \, dx$

206. $\int \frac{e^{\operatorname{arc\,tg} x} + \ln (1 + x^2) + 1}{1 + x^2} \, dx$

207. $\int \operatorname{arc\,sin} \sqrt{x/2} \, dx$

208. $\int \frac{\operatorname{arc\,sin} \sqrt{x}}{\sqrt{1 - x}} \, dx$

209. $\int x \operatorname{arc\,sin} x^2 \, dx$

210. $\int x (\operatorname{arc\,tg} x)^2 \, dx$

211. $\int \cos^2 (\ln x) \, dx$

212. $\int (x^2 - 2x + 3) \ln x \, dx$

213. $\int 3^x \cos 3x \, dx$

214. $\int \ln (x + \sqrt{1 + x^2}) \, dx$

215. $\int x \, 2^{-x} \, dx$

216. $\int \sqrt{\frac{\operatorname{arc\,sin} x}{1 - x^2}} \, dx$

217. $\int \frac{dx}{\sqrt{(1 + x^2) \ln (x + \sqrt{1 + x^2})}}$

218. $\int \frac{x}{\sqrt{(1 - x^2)^3}} \operatorname{arc\,sin} x \, dx$

219. $\int \sqrt{\frac{\ln (x + x^2 + 1)}{1 + x^2}} \, dx$

220. $\int \frac{x e^x \, dx}{(1 + x)^2}$

221. $\int \frac{x - \sqrt{\operatorname{arc\,tg} 2x}}{1 + 4x^2} \, dx$

222. $\int \frac{x \operatorname{arc\,tg} x}{(1 + x^2)^2} \, dx$

223. $\int \frac{\operatorname{arc\,sin} x}{x^2} \, dx$

□ BAB II
METODE INTEGRASI

1. INTEGRAL DARI BENTUK FUNGSI GONIOMETRI

Rumus-rumus dasar dalam goniometri yang banyak dipakai, sebagai pembantu dalam menyelesaikan integral, adalah :

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \text{tg}^2 x = \sec^2 x$
3. $1 + \text{ctg}^2 x = \text{cosec}^2 x$
4. $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
5. $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
6. $\sin x \cos x = \frac{1}{2} \sin 2x$
7. $\sin x \cos y = \frac{1}{2}[\sin (x - y) + \sin (x + y)]$
8. $\sin x \sin y = \frac{1}{2}[\text{cps } (x - y) - \cos (x + y)]$
9. $\cos x \cos y = \frac{1}{2}[\cos (x - y) + \cos (x + y)]$
10. $1 - \cos x = 2 \sin^2 \frac{1}{2} x$
11. $1 + \cos x = 2 \cos^2 \frac{1}{2} x$
12. $1 \pm \sin x = 1 \pm \cos (\frac{1}{2}\pi - x)$

a. Integral dari bentuk

$$\int \sin^m x \cos^n x \, dx$$

dimana m dan n bulat.

(i). Andaikan m bulat positif dan ganjil, misal $m = 2k + 1$, maka :

$$\sin^m x \cos^n x = \sin^{2k} x \cos^n x \sin x$$

sedang $\sin^{2k} x = (1 - \cos^2 x)^k$

J a d i :

$$\sin^m x \cos^n x \, dx = \sin^{2k} x \cos^n x \sin x \, dx$$

$$= - (1 - \cos^2 x)^k \cos^n x \, d(\cos x)$$

(ii). Andaikan n bulat positif dan ganjil, misal $n = 2k + 1$, maka :

$$\sin^m x \cos^n x = \sin^m x \cos^{2k} x \cos x$$

sedang $\cos^{2k} x = (1 - \sin^2 x)^k$

J a d i :

$$\int \sin^m x \cos^n dx = \int \sin^m x \cos^{2k} x \cos dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \, d(\sin x)$$

Agar lebih jelas, kita lihat beberapa contoh soal.

1). Tentukan : $\int \sin^2 x \cos^2 x \, dx$

Penyelesaian :

Disini cos x mempunyai pangkat ganjil, maka :

$$\cos^2 x = \cos^2 x \cos x = (1 - \sin^2 x). \cos x$$

J a d i :

$$\int \sin^2 x \cos^2 x \, dx = \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int \sin^2 x - \sin^4 x \, d(\sin x)$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C.$$

2). Tentukan : $\int \cos^4 2x \sin^3 2x \, dx.$

Penyelesaian :

Disini sin 2 x mempunyai pangkat ganjil, maka :

J a d i :

$$\sin^3 2x = \sin^2 2x \sin 2x = (1 - \cos^2 2x) \sin 2x$$

$$\int \cos^4 2x \sin^3 2x \, dx = \int \cos^4 2x (1 - \cos^2 2x) \sin 2x \, dx$$
$$= -\tfrac{1}{2}\int [\cos^4 2x - \cos^6 2x] \, d(\cos 2x)$$
$$= \frac{1}{14}\cos^7 2x - \frac{1}{10}\cos^5 2x + C.$$

3). Tentukan : $\int \sin^3 3x \cos^5 3x \, dx$.

Penyelesaian :

Oleh karena kedua suku dalam integran mempunyai pangkat ganjil, maka kita rubah suku yang mempunyai pangkat terkecil.

J a d i :

$$\int \sin^3 3x \cos^5 3x \, dx = \int (1 - \cos^2 3x) \cos^5 3x \sin 3x \, dx$$
$$= -\frac{1}{3}\int (1 - \cos^2 3x) \cos^5 3x \, d(\cos 3x)$$
$$= \frac{1}{24}\cos^8 3x - \frac{1}{18}\cos^6 3x + C.$$

4). Tentukan : $\int \cos^3 x \, dx$.

Penyelesaian :

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \, d(\sin x)$$
$$= \sin x - \frac{1}{3}\sin^3 x + C.$$

5). Tentukan : $\int \sin^5 x \, dx$

Penyelesaian :

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = -\int (1 - \cos^2 x)^2 \, d(\cos x)$$
$$= -\int (1 - 2\cos^2 x + \cos^4 x) \, d(\cos x)$$
$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.$$

(iii). Jika m dan n bulat positif dan genap, maka integran dari bentuk :

$$\int \sin^m x \cos^n x \, dx$$

dirubah dengan menggunakan rumus-rumus :

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

Contoh-contoh soal :

6). Tentukan : $\int \cos^2 3x \sin^4 3x \, dx$.

Penyelesaian :

$$\int \cos^2 3x \sin^4 x \, dx = \int (\cos 3x \sin 3x)^2 \sin^2 3x \, dx$$

$$= \int \frac{\sin^2 6x}{4} \cdot \frac{1 - \cos 6x}{2} \, dx$$

$$= \frac{1}{8} \int (\sin^2 6x - \sin^2 6x \cos 6x) \, dx$$

$$= \frac{1}{8} \int (\frac{1 - \cos 12x}{2} - \sin^2 6x \cos 6x) \, dx$$

$$= \frac{1}{8} (\frac{x}{2} - \frac{\sin 12x}{24} - \frac{1}{18} \sin^3 6x) + C.$$

7). Tentukan : $\int \cos^4 x \, dx$

Penyelesaian :

$$\int \cos^4 x \, dx = \frac{1}{4} \int (1 + \cos 2x)^2 \, dx$$

$$= \frac{1}{4} \int (1 + 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{8} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{4} x + \frac{1}{4} \sin 2x + \frac{1}{8} x + \frac{1}{32} \sin 4x + C.$$

(iv). Jika m dan n bilangan bulat negatif, misal $m = -k$ dan $n = -h$, maka :

$$\int \sin^m x \cos^n x \, dx = \int \frac{dx}{\sin^k x \cos^h x} = \int \operatorname{cosec}^k x \sec^h x \, dx$$

$$= \int \operatorname{cosec}^k x \sec^{h-2} x \, d(\operatorname{tg} x)$$

$$= \int (1 + \frac{1}{\operatorname{tg}^2 x})^{k/2} (1 + \operatorname{tg}^2 x)^{(h-2)/2} d(\operatorname{tg} x)$$

$$= \int \frac{(1 + \operatorname{tg}^2 x)^{\frac{k+h}{2} - 1}}{\operatorname{tg}^k x} d(\operatorname{tg} x)$$

Dalam keadaan khusus, integral diatas direduksi menjadi bentuk :

$$\int \frac{dx}{\sin^k x} = \frac{1}{2^{k-1}} \int \frac{d\left(\frac{x}{2}\right)}{\sin^k \frac{x}{2} \cos^k \frac{x}{2}} \qquad \text{dan}$$

$$\int \frac{dx}{\cos^h x} = \int \frac{d\left(x + \frac{\pi}{2}\right)}{\sin^h \left(x + \frac{\pi}{2}\right)}$$

Contoh - contoh soal

9). Tentukan : $\int \frac{dx}{\sin^2 2x \cos^4 2x}$

Penyelesaian :

$$\int \frac{dx}{\sin^2 2x \cos^4 2x} = \int \operatorname{cosec}^2 2x \sec^4 2x \, dx$$

$$= \tfrac{1}{2} \int \left(1 + \frac{1}{\operatorname{tg}^2 2x}\right) \cdot (1 + \operatorname{tg}^2 2x) \, d(\operatorname{tg} 2x)$$

$$= \tfrac{1}{2} \int \left(2 + \operatorname{tg}^2 2x + \frac{1}{\operatorname{tg}^2 2x}\right) d(\operatorname{tg} 2x)$$

$$= \operatorname{tg} 2x + \frac{1}{6} \operatorname{tg}^3 2x - \frac{1}{2 \operatorname{tg} 2x} + C.$$

10). Tentukan : $\int \frac{dx}{\cos^4 x}$

Penyelesaian :

$$\int \frac{dx}{\cos^4 x} = \int \sec^2 x \, d(\operatorname{tg} x) = \int (1 + \operatorname{tg}^2 x) \, d(\operatorname{tg} x)$$

$$= \operatorname{tg} x + \frac{1}{3} \operatorname{tg}^3 + C.$$

11). Tentukan : $\int \frac{dx}{\sin^3 x}$

Penyelesaian :

$$\int \frac{dx}{\sin^3 x} = \frac{1}{2^3} \int \frac{dx}{\sin^3 \frac{x}{2} \cos^3 \frac{x}{2}} = \frac{1}{8} \int \operatorname{tg}^{-3} \frac{x}{2} \sec^6 \frac{x}{2} \, dx$$

$$= \frac{1}{8} \int \frac{(1 + \operatorname{tg}^2 x/2)^2}{\operatorname{tg}^3 x/2} \sec^2 x/2 \, dx.$$

$$= \frac{2}{8}\int (tg^{-3}\ x/2 + \frac{2}{tg\ x/2} + tg\ x/2)\ d\ (tg\ x/2)$$

$$= \frac{1}{4}\left[- \frac{1}{2\ tg^2\ x/2} + 2\ ln\ tg\ x/2 + \frac{tg^2\ x/2}{2}\right] + C.$$

b. Integral dari bentuk

$$\int tg^m\ x\ sec^n\ x\ dx$$

dan

$$\int cotg^m\ x\ cosec^n\ x\ dx$$

dimana m dan n bulat, positif.

Bentuk integral dapat dirubah dengan menggunakan rumus :

$tg^2\ x = sec^2\ x - 1$ atau $ctg^2\ x = cosec^2\ x - 1$

Contoh-contoh soal

12. Tentukan : $\int ctg^3\ 2\ x\ cosec\ 2\ x\ dx$

Penyelesaian :

$$\int ctg^3\ 2\ x\ cosec\ 2\ x\ dx = \int ctg^2\ 2\ x\ cosec\ 2\ x\ ctg\ 2\ x\ dx$$

$$= -\tfrac{1}{2}\int (cosec^2\ 2\ x - 1)\ d\ (cosec\ 2\ x)$$

$$= \tfrac{1}{2}\ cosec\ 2\ x - \frac{1}{6}\ cosec^3\ 2\ x + C.$$

13. Tentukan : $\int tg^3\ x\ sec^5\ x\ dx$

Penyelesaian :

$$\int tg^3\ x\ sec^5\ x\ dx = \int tg^2\ x\ sec^4\ x\ (tg\ x\ sec\ x)\ dx$$

$$= \int (sec^2\ x - 1)\ sec^4\ x\ d\ (sec\ x)$$

$$= \frac{1}{7}\ sec^7\ x - \frac{1}{5}\ sec^5\ x + C.$$

14. Tentukan : $\int tg^3\ 3\ x\ sec^4\ 3\ x\ dx$

Penyelesaian :

$$\int tg^3\ 3\ x\ sec^4\ 3\ x\ dx = \int tg^3\ 3\ x\ sec^2\ 3\ x\ sec^2\ 3\ x\ dx$$

$$= \frac{1}{3}\int tg^3\ 3x\ (tg^2\ 3x + 1)\ d\ (tg\ 3x)$$

$$= \frac{1}{18}\ tg^6\ 3x + \frac{1}{12}\ tg^4\ 3x + C.$$

15. Tentukan : $\int tg^3\ x\ dx$

Penyelesaian :

$$\int tg^3\ x\ dx = \int (sec^2\ x - 1)\ tg\ x\ dx = \int sec^2\ x\ tg\ x\ dx - \int tg\ x\ dx$$

$$= \tfrac{1}{2}\ tg^2\ x + \ln \cos x + C.$$

16. Tentukan : $\int ctg^4\ x\ dx.$

Penyelesaian :

$$\int ctg^4\ x\ dx = \int ctg^2\ x\ (cosec^2\ x - 1)\ dx$$

$$= \int ctg^2\ x\ cosec^2\ x\ dx - \int ctg^2\ x\ dx$$

$$= -\frac{1}{3}\ ctg^3\ x - \int (cosec^2\ x - 1)\ dx$$

$$= -\frac{1}{3}\ ctg^3\ x + ctg\ x + x + C.$$

17. Tentukan : $\int sec^4\ 2x\ dx$

Penyelesaian :

$$\int sec^4\ 2x\ dx = \int sec^2\ 2x\ (1 + tg^2\ 2x)\ dx$$

$$= \tfrac{1}{2}\int (1 + tg^2\ 2x)\ d\ tg\ 2x$$

$$= \tfrac{1}{2}\ tg\ 2x + \frac{1}{6}\ tg^3\ 2x + C.$$

18. Tentukan : $\int cosec^6\ x\ dx$

Penyelesaian :

$$\int cosec^6\ x\ dx = \int (ctg^2\ x + 1)^2\ cosec^2\ x\ dx$$

$$= -\int (ctg^4\ x + 2\ ctg^2\ x + 1)\ dctg\ x$$

$$= -\left(\frac{1}{5}\ ctg^5\ x + \frac{2}{3}\ ctg^3\ x + ctg\ x\right) + C$$

c. Integral dari bentuk

$\int \sin mx \cos nx\ dx$; $\int \sin mx \sin nx\ dx$; $\int \cos mx \cos nx\ dx$

Didalam soal - soal ini digunakan rumus - rumus :

1). $\sin mx \cos nx = \tfrac{1}{2}\left[\sin (m + n)\ x + \sin (m - n)\ x\right]$;

2). $\sin mx \sin nx = \frac{1}{2}\left[\cos (m - n) x - \cos (m + n) x\right]$;

3). $\cos mx \cos nx = \frac{1}{2}\left[\cos (m - n) x + \cos (m + n) x\right]$

Contoh - contoh soal :

19. Tentukan : $\int \sin 9x \sin x \, dx$

Penyelesaian :

$$\int \sin 9x \sin x \, dx = \frac{1}{2}\int \left[\cos 8x - \cos 10x\right] dx$$
$$= \frac{1}{16} \sin 8x - \frac{1}{20} \sin 10x + C.$$

20. Tentukan : $\int \sin 3x \cos 5x \, dx$

Penyelesaian :

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int \left[\sin (3 - 5) x + \sin (3 + 5) x\right] dx$$
$$= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx$$
$$= \frac{1}{4} \cos 2x - \frac{1}{16} \cos 8x + C.$$

21. Tentukan : $\int \cos 8x \, 2x \, dx$

Penyelesaian :

$$\int \cos 8x \cos 2x \, dx = \frac{1}{2}\int \left[\cos (8 - 2) x + \cos (8 + 2) x\right] dx$$
$$= \frac{1}{12} \sin 6x + \frac{1}{20} \sin 10x + C .$$

22. Tentukan : $\int (1 + \cos 5x)^{3/2} \, dx.$

Penyelesaian :

$$\int (1 + \cos 5x)^{3/2} \, dx = \int (2 \cos^2 \tfrac{5}{2} x)^{3/2} \, dx$$
$$= 2\sqrt{2} \int (\cos \tfrac{5}{2} x)^3 \, dx = 2\sqrt{2} \int (1 - \sin^2 \tfrac{5}{2} x) \cos \tfrac{5}{2} x \, dx$$
$$= \frac{4}{5}\sqrt{2} \left[\sin \tfrac{5}{2} x - \frac{1}{3} \sin^3 \tfrac{5}{2} x\right] + C.$$

23. Tentukan : $\int \frac{dx}{\sqrt{1 - \sin 4x}}$

Penyelesaian :

$$\int \frac{dx}{\sqrt{1 - \sin 4x}} = \int \frac{dx}{\sqrt{1 - \cos (\frac{\pi}{2} - 4x)}} = \frac{1}{2}\sqrt{2} \int \frac{dx}{\sin (\frac{\pi}{4} - 2x)}$$

$$= -\frac{\sqrt{2}}{2} \ln\left[\cos\left(\frac{\pi}{4} - 2x\right) - \operatorname{ctg}\left(\frac{\pi}{4} - 2x\right)\right] + C.$$

24. Tentukan : $\int \sin 3x \sin 5x \, dx$

Penyelesaian :

$$\int \sin 3x \sin 5x \, dx = \tfrac{1}{2} \int (\cos 2x - \cos 8x) \, dx$$

$$= \frac{1}{4} \sin 2x - \frac{1}{16} \sin 8x + C.$$

Soal - soal :

224. $\int \operatorname{ctg}^5 3x \, dx$

225. $\int \operatorname{cosec}^8 2x \, dx$

226. $\int \operatorname{tg}^3 \frac{x}{3} \sec^3 \frac{x}{3} \, dx$

227. $\int (\operatorname{ctg}^2 2x + \operatorname{ctg}^4 2x) \, dx$

228. $\int \operatorname{tg}^3 \frac{5}{2} x \sec \frac{5}{2} x \, dx$

229. $\int \operatorname{ctg}^2 2x \operatorname{cosec}^4 2x \, dx$

230. $\int \frac{\sec^4 x}{\sqrt{\operatorname{tg} x}} \, dx$

231. $\int \left[\frac{\operatorname{cosec} bx}{\operatorname{tg} bx}\right]^3 dx$

232. $\int \frac{\operatorname{tg}^5 2x \, dx}{\sec^3 2x}$

233. $\int \frac{dx}{\sin^2 x \cos^2 x}$

234. $\int \cos^7 3x \, dx$

235. $\int \sec^6 x \, dx$

236. $\int \operatorname{tg}^n \sec^4 x \, dx$

237. $\int (\operatorname{tg} bx - \operatorname{ctg} bx)^3 \, dx$

238. $\int \operatorname{tg}^{3/2} x \sec^4 x \, dx$

239. $\int \frac{\operatorname{cosec} ax^4}{\operatorname{ctg} ax} \, dx$

240. $\int \sin^5 x^3 \operatorname{Cos} x \, dx$

241. $\int \frac{\sec^4 x}{\operatorname{tg}^3 x} \, dx$

242. $\int \left(\frac{\operatorname{tg} \emptyset}{\operatorname{ctg} \emptyset}\right)^3 d\emptyset$

243. $\int \frac{dx}{\sin^4 3x \cos^2 3x}$

244. $\int \frac{\cos^5 x}{\sin^{2/3} x} \, dx$

245. $\int \operatorname{tg} 5x \sec^7 5x \, dx$

246. $\int \cos^7 3x \, dx$

247. $\int \cos x \cos 7x \, dx$

248. $\int x \sin 2x \cos 2x \, dx$

249. $\int \cos 3x \cos 7x \, dx$

250. $\int \operatorname{ctg}^3 x \operatorname{cosec}^4 x \, dx$

251. $\int \frac{\cos^3 x \, dx}{1 - \sin x}$

252. $\int \frac{tg^3 x \, dx}{\sqrt{\sec x}}$

253. $\int \frac{\cos^4 x}{\sin^6 x} dx$

254. $\int \frac{\cos 4x + 2 \cos 2x}{\cos^2 x} dx$

255. $\int \sin x \, dx$

256. $\int \cos^2 x \sin x \, dx$

257. $\int \sin 2x \cos 4x \, dx$

258. $\int x (\cos^3 x^2 - \sin^3 x^2) dx$

259. $\int \sin^4 x \sec^2 x \, dx$

260. $\int \frac{ctg^3 x \, dx}{\text{cosec } x}$

261. $\int tg \, x \sqrt{\sec x} \, dx$

262. $\int \frac{\cos^2 x \, dx}{\sin^5 x}$

263. $\int (\frac{\sec ax}{tg \, ax})^4 dx$

264. $\int \frac{dx}{\sin^5 x \cos^3 x}$

265. $\int \frac{dx}{\sqrt{\sin x \cos^3 x}}$

2. INTEGRAL DENGAN MENGGUNAKAN SUBSTITUSI

Bila integral tak tertentu $\int f(x) \, dx$ tidak dapat langsung di-integral dengan menggunakan rumus-rumus atau cara yang telah dibicarakan dimuka, maka kita rubah bentuk integrannya ke suatu bentuk dengan jalan mengganti perubah x.

Perubah x atau fungsi dari perubah x diganti dengan suatu fungsi yang mempunyai perubah baru, misalnya perubah u atau t, sedemikian sehingga dapat di integralkan dengan cara-cara yang sudah diketahui. Tentu saja, dx juga harus diganti dalam bentuk suku dari u dan du.

Andaikan diambil :

$$x = \varphi(U) \longrightarrow dx = \varphi'(U) \, dU$$

Maka integral tak tertentu diatas menjadi :

$$\int f(x) \, dx = \int f[\varphi(U)] \, \varphi'(U) \, dU = \int \Psi(U) \, dU$$

$$= F(U) + C.$$

Oleh karena bentuk integral tak tertentu adalah dari fungsi x, maka pada hasil yang terakhir, perubah u harus dikembalikan ke fungsi dengan perubah x.

Mengganti perubah dalam suatu integral tak tertentu, dilakukan sesuai dengan bentuk integrannya.

a. Substitusi fungsi aljabar

(i) Jika integran memuat pangkat pecahan dari bentuk a + bx.

Diambil substitusi, untuk a + bx suatu U yang mempunyai pangkat, sehingga sesuai dengan a + bx.

Contoh - contoh soal.

1). Tentukan : $\int \frac{x^2}{(2 + 3x)^{2/3}}$

Penyelesaian :

Pangkat pecahan dari 2 + 3 x adalah $\frac{1}{3}$, maka diambil $U^3 = 2 + 3x$

dan $U^3 = (2 + 3x)^{2/3}$

$$3U^2\, dU = 3\, dx \longrightarrow dx = U^2\, dU.$$

Dari $U^3 = 2 + 3x \longrightarrow x = \frac{1}{3}(U^3 - 2)$

Jadi :

$$\int \frac{x^2\, dx}{(2 + 3)^{2/3}} = \frac{1}{9}\int \frac{(U^3 - 2)^2 . U^2\, dU}{U^2} = \frac{1}{9}\int (U^6 - 4U^3 + 4)\, dU$$

$$= \frac{1}{63}U^7 - \frac{1}{9}U^4 + \frac{4}{9}U + C.$$

Untuk mendapatkan hasil integrasi yang terakhir, perubah U diganti sesuai dengan harganya dalam suku dari x, yaitu :

$$U = (2 + 3x)^{1/3}$$

M a k a :

$$\int \frac{x^2\, dx}{(2 + 3x)^{2/3}} = \frac{1}{63}(2 + 3x)^{7/3} - \frac{1}{9}(2 + 3x)^{4/3} + \frac{4}{9}(2 + 3x)^{1/3} + C.$$

$$= \frac{(2 + 3x)^{1/3}}{63}(2 + 3x)^2 - 7(2 + 3x) + 28 + C.$$

$$= \frac{(2 + 3x)^{1/3}}{63} (9x^2 - 9x + 18) + C.$$

$$= \frac{1}{7} (2 + 3x)^{1/3} (x^2 - x + 2) + C.$$

2). Tentukan : $\int \frac{dx}{4 + \sqrt{4 + 5x}}$

Penyelesaian :

Andaikan : $U^2 = 4 + 5x \longrightarrow 2U\,dU = 5\,dx$

$dx = \frac{2}{5} U\,dU$

Bentuk integral sekarang :

$$\int \frac{dx}{4 + \sqrt{4 + 5x}} = \frac{2}{5} \int \frac{U\,dU}{4 + U} = \frac{2}{5} \int \left(1 - \frac{4}{U + 4}\right) dU$$

$$= \frac{2}{5} \left[U - 4 \ln (U + 4)\right] + C.$$

Jadi hasil integral :

$$\int \frac{dx}{4 + \sqrt{4 + 5x}} = \frac{2}{5} \sqrt{4 + 5x} - \frac{8}{5} \ln (4 + \sqrt{4 + 5x}) + C.$$

3). Tentukan : $\int \frac{\sqrt{9 - 4x}}{x} dx$

Penyelesaian :

Andaikan : $U^2 = 9 - 4x \longrightarrow 2U\,dU = -4\,dx$

$dx = -\frac{1}{2} U\,dU.$

Sedang : $x = \frac{9 - U^2}{4}$

Maka :

$$\int \frac{\sqrt{9 - 4x}}{x} dx = -2 \int \frac{U^2\,dU}{9 - U^2} = 2 \int \left(1 + \frac{9}{U^2 - 9}\right) dU$$

$$= 2 \left[U + \frac{9}{6} \ln \left| \frac{U - 3}{U + 3} \right| \right] + C.$$

Jadi :

$$\int \frac{\sqrt{9 - 4x}}{x}\, dx = 2\left[\sqrt{9 - 4x} + 3 \ln \left| \frac{3 - \sqrt{9 - 4x}}{3 + \sqrt{9 - 4x}} \right| \right] + C.$$

4). Tentukan : $\int \frac{x\, dx}{\sqrt{x + 3}}$

Penyelesaian :

Andaikan : $U^2 = x + 3 \longrightarrow 2U\, dU = dx$

$x = U^2 - 3.$

M a k a :

$$\int \frac{x\, dx}{\sqrt{x + 3}} = 2\int \frac{(U^2 - 3)\, U\, dU}{U} = 2\int (U^2 - 3)\, dU$$

$$= \frac{2}{3} U^3 - 6U + C.$$

J a d i :

$$\int \frac{x\, dx}{\sqrt{x + 3}} = \frac{2}{3}\sqrt{(x + 3)^3} - 6\sqrt{x + 3} + C.$$

(ii). Jika Integran memuat pangkat pecahan dari bentuk $a + bx^n$.

Diambil substitusi, untuk $a + bx^n$ suatu u yang mempunyai-pangkat sehingga sesuai dengan $a + bx^n$.

Contoh - contoh soal :

5). Tentukan : $\int \frac{(x^2 - a^2)^{3/2}}{x}\, dx$

Penyelesaian :

Andaikan : $U^2 = x^2 - a^2 \longrightarrow 2U\, dU = 2x\, dx.$ yang kita butuhkan bukan x dx, akan tetapi $\frac{dx}{x}$.

M a k a : $\frac{dx}{x} = \frac{U\, dU}{U^2 + a^2}$, selanjutnya :

$$\int \frac{(x^2 - a^2)^{3/2}}{x}\, dx = \int U^3 \cdot \frac{U}{U^2 + a^2}\, dU = \int \frac{U^4\, dU}{U^2 + a^2}$$

$$= \int \left(U^2 - a^2 + \frac{a^4}{U^2 + a^2}\right) dU$$

$$= \frac{1}{3} U^3 - a^2 U + a^3 \text{ arc tg } \frac{U}{a} + C.$$

Jadi hasil integralnya :

$$\int \frac{(x^2 - a^2)^{3/2}}{x}\,dx = \frac{1}{3}(x^2 - a^2)^{3/2} - a^2\sqrt{x^2 - a^2} + \text{arc tg}\,\frac{\sqrt{x^2 - a^2}}{a} + C.$$

6). Tentukan : $\int x^5 \sqrt{2 + x^3}\,dx$

Penyelesaian :

Andaikan : $U^2 = x^3 + 2 \longrightarrow 2\,U\,dU = 3\,x^2\,dx$

M a k a : $x^5\,dx = \frac{2}{3}\,U\,(U^2 - 2)\,du$

Sekarang :

$$\int x^5 \sqrt{2 + x^3}\,dx = \frac{2}{3}\int U^2\,(U^2 - 2)\,dU$$

$$= \frac{2}{3}\int (U^4 - 2\,U^2)\,dU = \frac{2}{15}\,U^5 - \frac{4}{9}\,U^3 + C.$$

$$= \frac{2}{45}\,(3\,U^5 - 10\,U^3) + C$$

$$= \frac{2}{45}\,U^3\,(3\,U^2 - 10) + C$$

$$= \frac{2}{45}\,(x^3 + 2)^{3/2}\left[(3\,(x^3 + 2) - 10\right] + C.$$

$$= \frac{2}{45}\,(x^3 + 2)^{3/2}\,(3\,x^3 - 4) + C.$$

7). Tentukan : $\int \frac{x3\,dx}{(x^2 + 4)^{\frac{1}{2}}}$

Penyelesaian :

Andaikan : $U^2 = x^2 + 4 \longrightarrow 2\,U\,dU = 2\,x\,dx$

M a k a : $x^3\,dx = U\,(U^2 - 4)\,dU.$

Bentuk integral sekarang :

$$\int \frac{x^3 dx}{(x^2 + 4)^{\frac{1}{2}}} = \int \frac{(U^2 - 4)\,dU}{U^4} = \int \frac{U^3 - 4\,U}{U^4}\,dU$$

$$= \ln U + \frac{2}{U^2} + C.$$

J a d i :

$$\int \frac{x^3}{(x^2 + 4)^{\frac{1}{2}}}\,dx = \ln \sqrt{x^2 + 4} + \frac{1}{x^2 + 4} + C.$$

(iii). Binominal differensial

Bentuk differensial $x^m (a + bx^n)^{p/q} dx$, dimana

$q > p$ $n \neq 0$,

p dan q semua bilangan bulat disebut binominal differensial, (Sedang m dan n bilangan rasional).

Bentuk integral : $x^m (a + bx^n)^{p/q} dx$.

dapat diselesaikan, dengan menggunakan substitusi :

(i). Substitusi $U^q = a + bx^n$, bila $\frac{m+1}{n}$ = bulat atau nol

(ii). Substitusi $U^q x^n = a + bx^n$, bila $\frac{m+1}{n} + \frac{p}{q}$ = bulat atau nol.

Kedua syarat diatas ini disebut Syarat Chebyshev.

8). Tentukan : $\int \frac{\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt{x}} dx$

Penyelesaian :

Disini : $m = -\frac{1}{2}$; $n = \frac{1}{4}$; $\frac{p}{q} = \frac{1}{3}$; $\frac{m+1}{n} = \frac{-\frac{1}{2}+1}{\frac{1}{4}} = 2$

Maka diambil :

$$1 + x^{\frac{1}{4}} = U^3 \longrightarrow x = (U^3 - 1)^4$$

$$dx = 12\, U^2 (U^3 - 1)^3\, dU$$

J a d i :

$$\int \frac{\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt{x}} dx = 12 \int \frac{U^3 (U^3 - 1)^3}{(U^3 - 1)^2} dU = 12 \int (U^6 - U^3)\, dU$$

$$= 12 \left(\frac{U^7}{7} - \frac{U^4}{4}\right) + C$$

$$= \frac{12}{7} \left(\sqrt[3]{1 + \sqrt[4]{x}}\right)^7 - 3 \left(\sqrt[3]{1 + \sqrt[4]{x}}\right)^4 + C.$$

9). Tentukan : $\int \frac{x^3\, dx}{(a + bx^2)^{3/2}}$

Penyelesaian :

Disini : $m = 3$; $n = 2$; $p = -3$; $q = 2$, sedang

$$\frac{m+1}{n} = \frac{3+1}{2} = 2 \text{ (bulat). Maka diambil } -$$

substitusi : $a + bx^2 = U^2$, $x^2 = \frac{1}{b}(U^2 - a) \longrightarrow 2x\,dx = \frac{2}{b} U\,dU$

$$x^3\,dx = \frac{1}{b^2}(U^2 - a)\,U\,dU$$

J a d i :

$$\int \frac{x^3\,dx}{(a + bx^2)^{3/2}} = \frac{1}{b^2}\int \frac{(U^2 - a)\,U\,dU}{U^3} = \frac{1}{b^2}\int \left(1 - \frac{a}{U^2}\right) dU$$

$$= \frac{1}{b^2}\left(U + \frac{a}{U}\right) + C = \frac{1}{b^2}\left[\sqrt{a + bx^2} + \frac{a}{\sqrt{a + bx^2}}\right] + C.$$

10). Tentukan : $\int \frac{dx}{x^4\sqrt{1 + x^2}}$

Penyelesaian :

Disini : $m = -4$; $n = 2$, $\frac{p}{q} = -\frac{1}{2}$ dan

$$\frac{m+1}{n} + \frac{p}{q} = \frac{-4+1}{2} - \frac{1}{2} = -2 \text{ (bulat).}$$

Oleh karena itu, diambil :

$$1 + x^2 = U^2 x^2 \longrightarrow U = \frac{(1 + x^2)^{\frac{1}{2}}}{x}$$

$$x^2 = \frac{1}{U^2 - 1}\,,\quad x = \frac{1}{(U^2 - 1)^{\frac{1}{2}}}\,,\quad dx = -\frac{U\,dU}{(U^2 - 1)^{\frac{1}{2}}}$$

J a d i :

$$\int \frac{dx}{x^4\sqrt{1 + x^2}} = -\int \frac{\frac{U\,dU}{(U^2 - 1)^{3/2}}}{\frac{1}{(U^2 - 1)^2} \cdot \frac{U}{(U^2 - 1)^{\frac{1}{2}}}} = -\int (U^2 - 1)\,dU$$

$$= U - \frac{1}{3}U^3 + C = \frac{(1 + x^2)^{\frac{1}{2}}}{x} - \frac{1}{3}\,\frac{(1 + x^2)^{3/2}}{x^3} + C.$$

$$= \frac{(2x^2 - 1)(1 + x^2)^{\frac{1}{2}}}{3x^3} + C.$$

11). Tentukan : $\int \frac{dx}{(x^2 (16 + x^4)^{3/4}}$

Penyelesaian :

Disini : $m = -2$, $n = 4$, $p = -3$, $q = 4$ dan

$$\frac{m+1}{n} + \frac{p}{q} = \frac{-2+1}{4} - \frac{3}{4} = -1 \text{ (bulat)}$$

Olen karena itu diambil :

$$U^4 x^4 = 16 + x^4 \longrightarrow x^4 = \frac{16}{U^4 - 1}$$

$$x = \frac{2}{(U^4 - 1)^{5/4}} \longrightarrow dx = -\frac{2U^3}{(U^4 - 1)^{5/4}}$$

J a d i :

$$\int \frac{dx}{x^2 (16 + x^4)^{3/4}} = \int \frac{\frac{-2U^3 dU}{(U^4 - 1)^{5/4}}}{\frac{4}{(U^4 - 1)^{\frac{1}{2}}} \cdot \frac{8^3}{(U^4 - 1)^{3/4}}}$$

$$= -\frac{1}{16} \int dU = -\frac{1}{16} U + C.$$

$$= -\frac{(16 + x^4)^{\frac{1}{4}}}{16x} + C.$$

b. Substitusi dengan trigonometri

Jika integran memuat bentuk :

(i) $\sqrt{a^2 - b^2 x^2}$, diambil substitusi : $x = \frac{a}{b} \sin U$.

(ii) $\sqrt{a^2 + b^2 x^2}$, diambil substitusi : $x = \frac{a}{b} \text{tg } U$.

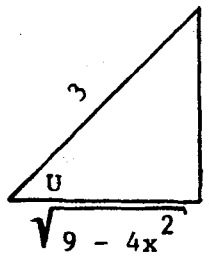
(iii) $\sqrt{b^2 x^2 - a^2}$, diambil substitusi : $x = \frac{a}{b} \sec U$.

Contoh - contoh soal :

1). Tentukan : $\int \frac{\sqrt{9 - 4x^2}}{x}\, dx$

Penyelesaian : Andaikan : $x = \frac{3}{2} \sin U \longrightarrow dx = \frac{3}{2} \cos U\, dU$

M a k a : $\int \frac{\sqrt{9 - 4x^2}}{x}\, dx = \int \frac{\sqrt{9 - 9\sin^2 U}}{\frac{3}{2}\sin U} \cdot \frac{3}{2} \cos U\, dU$



$= 3 \int \frac{\cos^2 U\, dU}{\sin U} = 3 \int \frac{1 - \sin^2(-U)}{\sin U} dU$

$= 3 \ln (\operatorname{cosec} U - \operatorname{ctg} U) + 3 \cos U + C.$

$= 3 \ln \frac{3 - \sqrt{9 - 4x^2}}{2x} + \sqrt{9 - 4x^2} + C.$

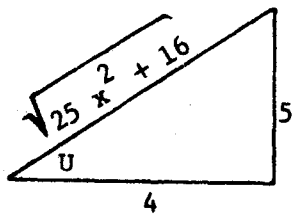
2). Tentukan : $\int \frac{dx}{x^2 \sqrt{25x^2 + 16}}$

Penyelesaian :

Andaikan : $x = \frac{4}{5} \operatorname{tg} U \longrightarrow dx = \frac{4}{5} \sec^2 U\, dU$

M a k a :

$$\int \frac{dx}{x^2 \sqrt{25x^2 + 16}} = \int \frac{\frac{4}{5} \sec^2 U\, dU}{(\frac{4}{5})^2 \operatorname{tg}^2 U \cdot 4 \sec U}$$



$= \frac{5}{16} \int \frac{\cos U\, dU}{\sin^2 U} = -\frac{5}{16 \sin U} + C.$

$= -\frac{\sqrt{25x^2 + 16}}{16x} + C.$

3). Tentukan : $\int \frac{x^2}{\sqrt{x^2 - 4}}\, dx.$

Penyelesaian :

Andaikan : $x = 2 \sec U \longrightarrow dx = 2 \sec U \operatorname{tg} U\, dU$

M a k a :

$$\int \frac{x^2}{\sqrt{x^2 - 4}}\, dx = \int \frac{4 \sec^2 U \cdot 2 \sec U \operatorname{tg} U\, dU}{2 \operatorname{tg} U}$$

$$= 4 \int \sec^3 U \, dU$$

$$= 2 \sec U \operatorname{tg} U + 2 \ln \left| \sec U + \operatorname{tg} U \right| + C$$

$$= \tfrac{1}{2} x \sqrt{x^2 - 4} + 2 \ln \left| x + \sqrt{x^2 - 4} \right| + C.$$

4). Tentukan : $\int \frac{x^2 \, dx}{\sqrt{2x - x^2}}$

Penyelesaian :

Pandang bentuk : $2x - x^2 = 1 - (x^2 - 2x + 1)$

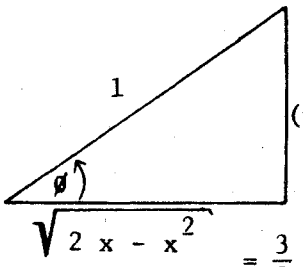
$= 1 - (x - 1)^2.$

Andaikan : $x - 1 = \sin \phi \longrightarrow dx = \cos \phi \, d\phi$

Sedang : $\sqrt{2x - x^2} = \sqrt{1 - (x - 1)^2} = \sqrt{1 - \sin^2 \phi} = \cos \phi$

M a k a :

$$\int \frac{x^2 \, dx}{\sqrt{2x - x^2}} = \int \frac{(1 + \sin \phi)^2}{\cos \phi} (\cos \phi \, d\phi)$$



$$= \int (1 + 2 \sin \phi + \sin^2 \phi) \, d\phi$$

$$= \int (1 + 2 \sin \phi + \tfrac{1}{2} - \tfrac{1}{2} \cos 2\phi) \, d\phi$$

$$= \tfrac{3}{2} \phi - 2 \cos \phi - \tfrac{1}{4} \sin 2\phi + C.$$

$$= \tfrac{3}{2} \arcsin (x - 1) - 2\sqrt{2x - x^2} - \tfrac{1}{2} (x - 1) \sqrt{2x - x^2} + C.$$

5). Tentukan : $\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}}$

Penyelesaian :

Pandang : $4x^2 - 24x + 27 = 4(x^2 - 6x) + 27 = 4(x - 3)^2 - 9$

Andaikan : $x - 3 = \frac{3}{2} \sec U \qquad dx = \frac{3}{2} \sec U \operatorname{tg} U \, dU$

M a k a :

$$\int \frac{dx}{(4x^2 - 24x + 27)^{3/2}} = \int \frac{\frac{3}{2} \sec U \operatorname{tg} U \, dU}{3^3 \operatorname{tg}^3 U}$$

$$= \frac{1}{18}\int \frac{\cos U \, dU}{\sin^2 U} = -\frac{1}{18} \sin^{-1} U + C.$$

$$= -\frac{x - 3}{9\sqrt{4x^2 - 24x + 27}} + C.$$

c. Integral dari fungsi pecah rasional

Suatu polinomial dalam x adalah suatu fungsi dengan bentuk $a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots + a_{n-1} + a_n$, dimana

a_i (i = 0, 1 , 2 , ..., n) adalah konstante dan n bulat positif dan termasuk nol.

Tiap-tiap polinomial dengan koefisien riil dapat dinyatakan sebagai perkalian dari faktor-faktor linier yang riil dalam bentuk ax + b dan / atau faktor-faktor kwadratis yang riil dalam bentuk $ax^2 + bx + c$.

Suatu fungsi $F(x) = \frac{f(x)}{g(x)}$, dimana f (x) dan g (x) adalah polinomial , disebut : FUNGSI PECAH RATIONAL.

Jika pangkat f (x) lebih rendah dari pada pangkat g (x), F (x) disebut PROPER ; untuk sebaliknya F (x) disebut IMPROPER.

Bentuk pecahan rasional yang improper dapat dinyatakan sebagai jumlahan dari suatu polinomial dan suatu pecahan rasional yang proper.

Sebagai contoh :

$$\frac{x^4 - x^3 - x - 1}{x^3 - x^2} = x - \frac{x + 1}{x^3 - x^2}$$

Setiap pecahan rasional yang proper dapat dinyatakan sebagai suatu jumlahan dari pecahan-pecahan yang sederhana yang penyebut - nya berbentuk :

$(ax + b)^n$ dan/atau $(ax^2 + bx + c)^n$, dimana n bulat positip. Kita tinjau 4 kemungkinan yang timbul dalam pecahan rasional yang proper :

(i). Semua faktor dari penyebut linier dan berlainan.

Jika pecahan rasional yang proper F (x), penyebut g (x) dapat dinyatakan sebagai perkalian faktor-faktor linier yang berlainan, misalnya :

$$g(x) = (x - a_1)(x - a_2) \ldots\ldots\ldots (x - a_n)$$

dimana : $a_1 \neq a_2 \neq a_3 \neq \ldots\ldots\ldots\ldots \neq a_n$, maka :

$$F(x) = \frac{f(x)}{g(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \frac{A_3}{x - a_3} + \ldots\ldots + \frac{A_n}{x - a_n}$$

Untuk menghitung A_1, A_2, $\ldots\ldots A_n$ kedua bagian diatas disamakan, atau mengambil harga-harga x tertentu. Jadi disini ada dua methode untuk menghitung koeffisien-koeffisien tak tertentu tersebut. Agar lebih jelas diambil beberapa contoh soal :

1). Tentukan : $\int \frac{2x + 1}{x^3 - 7x + 6}\, dx$

Penyelesaian :

Penyebut : $x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$

Oleh karena itu, pecahan rasional dapat ditulis :

$$\frac{2x + 1}{(x - 1)(x - 2)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x + 3}$$

$$= \frac{A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)}{(x - 1)(x - 2)(x + 3).}$$

Maka dipenuhi bentuk :

$$2x + 1 = A(x - 2)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x - 2)$$

Untuk mencari nilai A, B, dan C diambil dua methode.

Methode A.

Bagian kiri identik dengan bagian kanan, berarti koeffisien-koeffisien dari x yang berpangkat sama dari kedua bagian tersebut harus sama.

Jadi : koeffisien $x^2 \longrightarrow 0 = A + B + C$

koeffisien $x \longrightarrow 2 = A + 2B - 3C$

koeffisien $x^0 \longrightarrow 1 = -6A - 3B + 2C.$

Dari ketiga persamaan diatas, dapat dihitung nilai-nilai A , B dan C, yaitu : $A = -\frac{3}{4}$, B - 1 dan $C = -\frac{1}{4}$

Sekarang integral :

$$\int \frac{(2x+1)}{x^3 - 7x + 6}\,dx = -\frac{3}{4}\int \frac{dx}{x-1} + \int \frac{dx}{x-2} - \frac{1}{4}\int \frac{dx}{x+3}$$

$$= -\frac{3}{4}\ln(x-1) + \ln(x-2) - \frac{1}{4}\ln(x+3) + C$$

$$= \frac{1}{4}\ln\left|\frac{(x-2)^4}{(x-1)^3(x+3)}\right| + C$$

Methode B.

$2x + 1 = A(x-2)(x+3) + B(x-1)(x+3) + C(x-1)(x-2)$

Diambil harga x yang tertentu :

(1). Untuk x = 1 , $3 = -4A$ dan $A = -\frac{3}{4}$

(2). Untuk x = 2 , $5 = 5B$ dan $B = 1$

(3). Untuk x = - 3 , $-5 = 20C$ dan $C = -\frac{1}{4}$

Ternyata hasilnya sama dengan metode identitas.

2). Tentukan : $\int \frac{3x^2 + 32x - 88}{x^3 - x^2 - 14x + 24}\,dx$

Penyelesaian :

Penyebut : $x^3 - x^2 - 14x + 24 = (x-2)(x-3)(x+4)$

M a k a :

$$\frac{3x^2 + 32x - 88}{x^3 - x^2 - 14x + 24} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{x+4}$$

$$= \frac{A(x-3)(x+4) + B(x-2)(x+4) + C(x-2)(x-3)}{(x-2)(x-3)(x+4)}$$

Ternyata :

$$3x^2 + 32x - 88 = A(x-3)(x+4) + B(x-2)(x+4) + C(x-2)(x-3)$$

Digunakan metode B.

Untuk harga : $x = 2 \longrightarrow -12 = -6A$ dan $A = 2$

$x = 3 \longrightarrow 35 = 7B$ dan $B = 5$

$x = -4 \longrightarrow -168 = 42C$ dan $C = -4$

J a d i :

$$\int \frac{3x^2 + 32x - 88}{x^3 - x^2 - 14x + 24}\,dx = 2\int \frac{dx}{x-2} + 5\int \frac{dx}{x-3} - 4\int \frac{dx}{x+4}$$

$$= 2\ln(x-2) + 5\ln(x-3) - 4\ln(x+4) + C$$

$$= \ln \frac{(x-2)^2 (x-3)^5}{(x+4)^4} + C$$

3). Tentukan : $\int \frac{25x^2 - 20x + 2}{x^3 - 5x^2 + 4x}\,dx$

Penyelesaian :

Penyebut : $x^3 - 5x^2 + 4x = x(x-1)(x-4)$

M a k a :

$$\frac{25x^2 - 20x + 2}{x^3 - 5x^2 + 4x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-4}$$

$$= \frac{A(x-1)(x-4) + Bx(x-4) + Cx(x-1)}{x(x-1)(x-4)}$$

Ternyata :

$$25x^2 - 20x + 2 = A(x-1)(x-4) + Bx(x-4) + Cx(x-1)$$

Digunakan metode B.

Untuk $x = 0 \longrightarrow 2 = 4A$ dan $A = \frac{1}{2}$

$x = 1 \longrightarrow 7 = -3B$ dan $B = -\frac{7}{3}$

$x = 4 \longrightarrow 322 = 12C$ dan $C = \frac{161}{6}$

J a d i :

$$\int \frac{25x^2 - 20x + 2}{x^3 - 5x^2 + 4x}\,dx = \tfrac{1}{2}\int \frac{dx}{x} - \frac{7}{3}\int \frac{dx}{x-1} + \frac{161}{6}\int \frac{dx}{x-4}$$

$$= \tfrac{1}{2}\ln x - \frac{7}{3}\ln(x-1) + \frac{161}{6}\ln(x-4) + C.$$

(ii) Semua faktor dari penyebut linier, tetapi ada beberapa yang sama (berulang).

Untuk tiap faktor linier $ax + b$ yang timbul n kali dalam penyebut dari pecahan rasional, kita tulis sebagai jumlahan dari n pecahan persiil dalam bentuk :

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \ldots\ldots\ldots + \frac{A_n}{(ax+b)^n}$$

dimana A_i $(i = 1, 2, \ldots\ldots, n)$ konstante yang harus dicari.

Contoh - contoh soal :

4) Tentukan : $\int \frac{(3x^2 - 22x + 19)\,dx}{(x+2)(x-3)^2}$

Penyelesaian :

Pandang : $\frac{3x^2 - 22x + 19}{(x+2)(x-3)^2} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$

$$3x^2 - 22x + 19 = A(x-3)^2 + B(x+2)(x-3) + C(x+2)$$

Untuk $x = -2 \longrightarrow 75 = 25A$ dan $C = 3$.

Untuk $x = 3 \longrightarrow -20 = 5C$ dan $C = -4$.

Harga A dan C dimasukkan , kemudian dibawa kekiri.

$$3x^2 - 22x + 19 - 3(x-3)^2 + (x+2) = B(x+2)(x-3)$$

Ternyata $B = 0$.

M a k a :

$$\int \frac{(3x^2 - 22x + 19)\,dx}{(x+2)(x-3)^2} = 3\int \frac{dx}{x+3} - 4\int \frac{dx}{(x-3)^2}$$

$$= 3 \ln (x + 2) + \frac{4}{x - 3} + C.$$

5). Tentukan : $\int \frac{(x^2 - 2)\, dx}{x^3 (x + 2)^2}$

Penyelesaian :

$$\frac{x^2 - 2}{x^3 (x + 2)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x + 2} + \frac{E}{(x + 2)^2}$$

$$= \frac{A (x + 2)^2 x^2 = + B x (x + 2)^2 + C (x + 2)^2 +}{x^3 (x + 2)^2}$$

$$x^2 - 2 = Ax^2 (x + 2)^2 + Bx (x + 2)^2 + C (x + 2)^2 + Dx^3 (x + 2) + Ex^3$$

Untuk x = 0 $\longrightarrow$ $-2 = 4C$ dan $C = -\frac{1}{2}$

x = -2 $\longrightarrow$ $2 = -8E$ dan $E = -\frac{1}{4}$

Harga - harga ini dimasukkan, kemudian :

$$x^2 - 2 + \tfrac{1}{2} (x + 2)^2 + \tfrac{1}{4} x^3 = x (x + 2) Ax (x + 2) + B (x + 2) + Dx^2$$

$$\tfrac{1}{4} x (x + 2) (x + 4) = x (x + 2) Ax (x + 2) + B (x + 2) + Dx^2$$

$$\tfrac{1}{4} (x + 4) = Ax (x + 2) + B (x + 2) + Dx^2$$

Untuk x = 0 $\longrightarrow$ $1 = 2B$ dan $B = \frac{1}{2}$

x = -2 $\longrightarrow$ $\frac{1}{2} = 4D$ dan $D = \frac{1}{8}$

Sedang dengan metode A, terdapat bahwa :

$A + D = 0 \longrightarrow A = -\frac{1}{8}$,

J a d i :

$$\int \frac{(x^2 - 2)\, dx}{x^3 (x + 2)^2} = -\frac{1}{8} \int \frac{dx}{x} + \tfrac{1}{2} \int \frac{dx}{x^2} - \tfrac{1}{2} \int \frac{dx}{x^3} + \frac{1}{8} \int \frac{dx}{x + 2} - \tfrac{1}{4} \int \frac{dx}{(x + 2)^2}$$

$$= \frac{1}{8} \ln \left| \frac{x+2}{x} \right| - \frac{1}{2x} + \frac{1}{4x^2} + \frac{1}{4(x+2)} + C$$

(iii) Beberapa faktor penyebut adalah kwadratis dan tak berulang.

Untuk tiap-tiap faktor yang mempunyai bentuk : $ax^2 + bx + c$, dinyatakan sebagai pecahan persiil dari bentuk :

$$\frac{Ax + B}{ax^2 + bx + c} .$$

Contoh - contoh soal

6). Tentukan : $\int \frac{dx}{x^3 + x}$

Penyelesaian :

Pandang dulu bentuk :

$$\frac{1}{x^3 + x} = \frac{1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} = \frac{A(x^2 + 1) + (Bx + C)x}{x(x^2 + 1)}$$

M a k a :

$1 = A(x^2 + 1) + (Bx + C)x$, dengan menggunakan metode A

terdapat : $A = 1$, $B = -1$ dan $C = 0$.

J a d i :

$$\int \frac{dx}{x^3 + x} = \int \frac{dx}{x} - \int \frac{x}{x^2 + 1} dx$$

$$= \ln x - \tfrac{1}{2} \ln (x^2 + 1) + C = \tfrac{1}{2} \ln \left| \frac{x}{x^2 + 1} \right| + C.$$

7). Tentukan : $\int \frac{x^3 + x^2 + x + 2}{x^4 + 3x^2 + 2} dx$

Penyelesaian :

Penyebut : $x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$.

M a k a :

$$: \frac{x^3 + x^2 + x + 2}{(x^2 + 1)(x^2 + 2)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 2}$$

$$= \frac{(Ax + B)(x^2 + 2) + (Cx + D)(x^2 + 1)}{(x^2 + 1)(x^2 + 2)}$$

Ternyata : $x^3 + x^2 + x + 2 = (Ax + B)(x^2 + 2) +$

$+ (Cx + D)(x^2 + 1)$.

dengan menggunakan metode A :

Koeffisien $x^3 \longrightarrow 1 = A + C$; Koeffisien $x^2 \longrightarrow 1 = B + D$

Koeffisien $x^1 \longrightarrow 1 = 2A + C$; Koeffisien $x^o \longrightarrow 2 = 2B + D$

Disini ternyata ada empat persamaan, dengan 4 anu yang harus dicari. Maka : A = 0 , B = 1 , C = 1 dan D = 0.

J a d i :

$$\int \frac{x^3 + x^2 + x + 2}{(x^2 + 1)(x^2 + 2)}\, dx = \int \frac{dx}{x^2 + 1} + \int \frac{x\, dx}{x^2 + 2}$$

$$= \text{arc tg } x + \tfrac{1}{2} \ln (x^2 + 2) + C.$$

8). Tentukan : $\int \frac{7x^3 + 20x^2 + 35x - 13}{x^2 (x^2 + 4x + 13)}\, dx$

Penyelesaian :

$$\frac{7x^3 + 20x^2 + 35x - 13}{x^2 (x^2 + 4x + 13)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 4x + 13}$$

M a k a :

$7x^3 + 20x^2 + 35x - 13 = Ax(x^2 + 4x + 13) +$

$+ B(x^2 + 4x + 13) + x^2 (Cx + D)$.

Untuk x = 0 , terdapat $-13 = 13B$ dan $B = -1$.

$7x^3 + 20x^2 + 35x - 13 = Ax(x^2 + 4x + 13) - (x^2 + 4x + 13) +$

$+ x^2 (Cx + D)$.

$$7x^3 + 21x^2 + 39x = \left[A(x^2 + 4x + 13) + x(Cx + D)\right] x$$

$$7x^2 + 21x + 39 = A(x^2 + 4x + 13) + x(Cx + D).$$

Untuk $x = 0$, terdapat $39 - 13A$ dan $A = 3$

$$7x^2 + 21x + 39 = 3(x^2 + 4x + 13) + x(Cx + D),$$

$4x + 9 = Cx + D$, maka $C = 4$ dan $D = 9$.

J a d i :

$$\int \frac{7x^3 + 20x^2 + 35x - 13}{x^2(x^2 + 4x + 13)}\,dx = 3\int \frac{dx}{x} - \int \frac{dx}{x^2} + \int \frac{(4x + 9)\,dx}{x^2 + 4x + 13}$$

$$= 3 \ln x + \frac{1}{x} + 2\int \frac{(2x + 4)\,dx}{x^2 + 4x + 13} + \int \frac{dx}{(x + 2)^2 + 9}$$

$$= 3 \ln x + \frac{1}{x} + 2 \ln (x^2 + 4x + 13) + \frac{1}{3} \text{ arc tg } \frac{x + 2}{3} + C.$$

(iv). Beberapa faktor penyebut adalah kwadratis dan berulang.

Untuk faktor kwadratis dengan bentuk $ax^2 + bx + c$ yang berulang n kali dalam penyebut pada pecahan rasional yang proper, ditulis sebagai jumlahan dari n pecahan parsiil dalam bentuk :

$$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \ldots\ldots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

dimana A_i dan B_i konstante yang harus dicari.

Contoh - contoh soal :

9). Tentukan : $\int \frac{x^2 - x + 4}{(x^2 + 2x + 2)^2 (x - 1)}\,dx$

Penyelesaian :

Ditulis dulu :

$$\frac{x^2 - x + 4}{(x - 1)(x^2 + 2x + 2)^2} = \frac{A}{x - 1} + \frac{Bx + C}{x^2 + 2x + 2} + \frac{Dx + E}{(x^2 + 2x + 2)^2}$$

$$= \frac{A(x^2 + 2x + 2)^2 + (Bx + C)(x - 1)(x^2 + 2x + 2) + (Dx + E)(x - 1)}{(x - 1)(x^2 + 2x + 2)^2}$$

M a k a :

$$x^2 - x + 4 = A(x^2 + 2x + 2)^2 + (Bx + C)(x - 1)(x^2 + 2x + 2) +$$

$$+ (Dx + E)(x - 1)$$

Untuk $x = 1$, terdapat $4 = 25A$ dan $A = \frac{4}{25}$.

Setelah A dimasukkan dan dibawah kiri, terdapat bentuk :

$$\frac{1}{25}\left[25(x^2 - x + 4) - 4(x^2 + 2x + 2)^2\right] = (x - 1)\left[(Bx + C)(x^2 + 2x + 2) + (Dx + E)\right]$$

$$\frac{1}{25}(4x^2 + 20x^2 + 27x + 84) = Bx^3 + (2B + C)x^2 + (2B + 2C + D)x + 2C + E.$$

Jika digunakan identitas, terdapat :

$$B = -\frac{4}{25}$$

$$2B + C = -\frac{20}{25} \quad \text{dan} \quad C = -\frac{12}{25}$$

$$2B + 2C + D = -\frac{27}{25} \quad \text{dan} \quad D = \frac{5}{25} = \frac{1}{5}$$

$$2C + E = -\frac{84}{25} \quad \text{dan} \quad E = -\frac{60}{25} = -\frac{12}{5}.$$

J a d i , integral sekarang menjadi :

$$\int \frac{(x^2 - x + 4)\,dx}{(x - 1)(x^2 + 2x + 2)} = \frac{4}{25}\int \frac{dx}{x - 1} - \frac{1}{25}\int \frac{4x + 12}{x^2 + 2x + 2}\,dx + \frac{1}{5}\int \frac{x - 12}{(x^2 + 2x + 2)^2}\,dx$$

$$= \frac{4}{25} \ln (x - 1) - \frac{2}{25} \int \frac{2x + 2}{x^2 + 2x + 2} dx$$

$$- \frac{8}{25} \int \frac{dx}{(x + 1)^2 + 1} + \frac{1}{10} \int \frac{(2x + 2)\, dx}{(x^2 + 2x + 2)^2}$$

$$- \frac{26}{10} \int \frac{dx}{\left[(x + 1)^2 + 1\right]^2}$$

$$= \frac{4}{25} \ln (x - 1) - \frac{2}{25} \ln (x^2 + 2x + 2) - \frac{8}{25} \text{arc tg } (x + 1)$$

$$- \frac{1}{10 (x^2 + 2x + 2)} - \frac{26}{10} \int \frac{dx}{\left[(x + 1)^2 + 1\right]^2}$$

Pandang dulu bentuk integral :

$$\int \frac{dx}{\left[(x + 1)^2 + 1\right]^2}$$, substitusi: $x + 1 = \text{tg } \theta$,

$dx = \sec^2 \theta \, d\theta$

Untuk $(x + 1)^2 + 1 = \text{tg}^2 \theta + 1 = \sec^2 \theta$

M a k a :

$$\int \frac{dx}{\left[(x + 1)^2 + 1\right]^2} = \int \frac{\sec^2 \theta \, d\theta}{\sec^4 \theta} = \int \cos^2 \theta \, d\theta$$

$$= \tfrac{1}{2} \int (1 + \cos 2\theta) \, d\theta = \tfrac{1}{2} (\theta + \tfrac{1}{2} \sin 2\theta) + C$$

$$= \tfrac{1}{2} \text{ arc tg } (x + 1) + \frac{x + 1}{2 (x^2 + 2x + 2)} + C.$$

Jadi, hasil terakhir dari integral adalah :

$$\int \frac{(x^2 - x + 4)\, dx}{(x - 1)(x^2 + 2x + 2)^2} = \frac{4}{25} \ln (x - 1) - \frac{2}{25} \ln (x^2 + 2x + 2)$$

$$- \frac{81}{50} \text{ arc tg } (x + 1) - \frac{13x + 14}{10 (x^2 + 2x + 2)} + C.$$

10). Tentukan : $\int \frac{2x^3 + x + 3}{(x^2 + 2)^2}\, dx$

Penyelesaian :

Ditulis : $\frac{2x^3 + x + 3}{(x^2 + 2)^2} = \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2}$

M a k a : $2x^3 + x + 3 = (Ax + B)(x^2 + 2) + Cx + D$

Menggunakan metode A , terdapat :

koeffisien x^3 : $2 = A$; koeffisien x^2 : $0 = B$.

koeffisien x^1 : $1 = 2A + C$ dan $C = -3$

koeffisien x^o : $3 = 2B + D$ dan $D = 2$

Jadi bentuk integral :

$$\int \frac{2x^3 + x + 3}{(x^2 + 2)^2}\, dx = \int \frac{2x}{x^2 + 2}\, dx - \int \frac{3x - 2}{(x^2 + 2)^2}$$

$$= \ln (x^2 + 2) - 3\int \frac{x\, dx}{(x^2 + 2)^2} + 2\int \frac{dx}{(x^2 + 2)^2}$$

Diselesaikan dulu integral :

$\int \frac{dx}{(x^2 + 2)^2}$, misalkan : $x = \sqrt{2}\ tg\ \theta$, $dx = \sqrt{2}\ sec^2\ \theta\ d\theta$

$$x^2 + 2 = 2(tg^2\ \theta + 1) = 2\ sec^2\ \theta$$

$$\int \frac{dx}{(x^2 + 2)^2} = \int \frac{\sqrt{2}\ sec^2\ \theta}{4\ sec^4\ \theta}\, d\theta = \tfrac{1}{4}\sqrt{2}\int \cos^2\ \theta\, d\theta$$

$$= \tfrac{1}{4}\sqrt{2}\int \tfrac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{8}\sqrt{2}\,(\theta + \tfrac{1}{2}\sin 2\theta) + C - \frac{1}{8}\sqrt{2}\ arc\ tg\ \frac{x}{\sqrt{2}} +$$

$$+ \frac{x}{4(x^2 + 2)} + C.$$

Hasil integral seluruhnya :

$$\int \frac{2x^3 + x + 3}{(x^2 + 2)^2}\, dx = \ln (x^2 + 2) + \tfrac{1}{4}\sqrt{2}\ \text{arc tg}\ \frac{x}{\sqrt{2}} + \frac{x + 3}{2(x^2 + 2)} + C$$

d. Integral dari fungsi-fungsi irrasional

Integral bentuk :

$$\int F\left[X\ ,\ \left(\frac{ax + b}{ax - b}\right)^{p/q}\right] dx$$

dimana p dan q bilangan bulat.

Integral tersebut dapat diselesaikan dengan mengambil substitusi :

$$\frac{ax + b}{ax - b} = U^q$$

Contoh - contoh soal :

1). Tentukan : $\int \frac{1}{x}\sqrt{\frac{x + 4}{x - 1}}\ dx$

Penyelesaian :

Andaikan : $\frac{x + 4}{x + 1} = U^2$, $x + 4 = U^2 (x + 1)$

$$x = \frac{U^2 - 4}{1 - U^2} \longrightarrow dx = -\frac{6\ U\ dU}{(1 - U^2)^2}$$

Setelah dimasukkan , terdapat :

$$\int \frac{1}{x}\sqrt{\frac{x + 4}{x + 1}}\ dx = \int \frac{1 - U^2}{U^2 - 4} \cdot U \cdot -\frac{6\ U\ dU}{(1 - U^2)^2}$$

$$= 6 \int \frac{U^2\ dU}{(U^2 - 4)(U^2 - 1)}$$

$$= 8\int \frac{dU}{U^2 - 4} - 2\int \frac{dU}{U^2 - 1} = 2 \ln\left|\frac{U - 2}{U + 2}\right| - \ln\left|\frac{U - 1}{U + 1}\right| + C.$$

$$= 2 \ln\left|\frac{\sqrt{x + 4} - 2\sqrt{x + 1}}{\sqrt{x + 4} + 2\sqrt{x + 1}}\right| - \ln\left|\frac{\sqrt{x + 4} + \sqrt{x + 1}}{\sqrt{x + 4} + \sqrt{x + 1}}\right| + C.$$

2). Tentukan : $\int \sqrt[3]{\frac{x + 1}{x - 1}}\, dx$

Penyelesaian :

Andaikan : $\frac{x + 1}{x - 1} = z^3$, $x + 1 = z^3 (x - 1)$

$$x = \frac{z^3 + 1}{z^3 - 1}, \quad dx = \frac{3 z^2 (z^3 - 1) - 3 z^2 (z^3 + 1)}{(z^3 - 1)^2}\, dx$$

$$= - \frac{6 z^2}{(z^3 - 1)^2}\, dz$$

Sekarang integral adalah :

$$\int \sqrt[3]{\frac{x + 1}{x - 1}}\, dx = \int z \cdot - \frac{6 z^2}{(z^3 - 1)^2}\, dz = -\int \frac{6 z^3}{(z^3 - 1)^2}\, dz$$

Pandang dulu bentuk :

$$\frac{6 z^3}{(z^3 - 1)^2} = \frac{6 z^3}{(z - 1)^2 (z^2 + z + 1)^2}$$

$$= \frac{A}{(z - 1)^2} + \frac{B}{z - 1} + \frac{Cz + D}{(z^2 + z + 1)^2} + \frac{Ez + F}{z^2 + z + 1}$$

M a k a :

$$6 z^3 = A (z^2 + z + 1)^2 + B (z - 1) (z^2 + z + 1)^2 + (Cz + D)$$

$$(z - 1)^2 + (Ez + F) (z - 1)^2 (z^2 + z + 1).$$

Untuk $z = 1 \longrightarrow 6 = 3^2 A$ dan $A = \frac{2}{3}$. Jika dimasukkan terdapat :

$$6 z^3 - \frac{2}{3} (z^2 + z + 1)^2 = (z - 1)\Big[B (z^2 + z + 1)^2 + (Cz + D)$$

$$+ (Ez + F) (z - 1) (z^2 + z + 1)\Big]$$

$$-\frac{2}{3}(z^3 - 6z^2 - 3z - 1) = B(z^2 + z + 1)^2 + (Cz + D)(z - 1)$$

$$(Ez + F)(z - 1)(z^2 + z + 1).$$

Untuk $z = 1 \longrightarrow 6 = 3^2 B$ dan $B = \frac{2}{3}$. Jika dimasukkan terdapat:

$$-\frac{2}{3}(z^3 - 6z^2 - 3z - 1) - \frac{2}{3}(z^2 + z + 1)^2 = (z - 1)\left[(Cz + D) + (Ez + F)(z^2 + z + 1)\right].$$

$$-\frac{2}{3} z (z^2 + 4z + 1) = Cz + D + (Ez + F)(z^2 + z + 1)$$

Menurut identitas , terdapat :

koeffisien z^3 : $-\frac{2}{3} = E \longrightarrow E = -\frac{2}{3}$

koeffisien z^2 : $-\frac{8}{3} = E + F \longrightarrow F = -2$

koeffisien z^1 : $-\frac{2}{3} = C + E + F \longrightarrow C = 2$

koeffisien z^o : $0 = D + F \longrightarrow D = 2$

$$\int \frac{6z^3}{(z^3 - 1)^2}\,dz = \frac{2}{3}\int \frac{dz}{(z - 1)^2} + \frac{2}{3}\frac{dz}{z - 1} + \int \frac{2z + 1 + 1}{(z^2 + z + 1)^2}\,dz$$

$$-\frac{1}{3}\int \frac{2z + 1 + 5}{z^2 + z + 1}\,dz$$

$$= -\frac{2}{3}\cdot\frac{1}{z - 1} + \frac{2}{3}\ln(z - 1) - \frac{1}{z^2 + z + 1}$$

$$-\frac{1}{3}\ln(z^2 + z + 1) + \int \frac{dz}{(z^2 + z + 1)^2} - \frac{5}{3}\int \frac{dz}{(z + \frac{1}{2})^2 + \frac{3}{4}}$$

Dicari dulu bentuk :

$$\int \frac{dz}{\left[(z + \frac{1}{2})^2 + (\frac{1}{2}\quad 3)^2\right]^2}$$, misalkan : $z + \frac{1}{2} = \frac{1}{2}\sqrt{3}\ \text{tg}\ \emptyset$

$$dz = \frac{1}{2}\sqrt{3}\ \sec^2 \emptyset\ d\emptyset$$

Maka , integral menjadi :

$$\int \frac{\frac{1}{2}\sqrt{3}\sec^2\theta\, d\theta}{\frac{9}{4}\sec^4\theta} = \int \frac{2}{9}\sqrt{3}\cos^2\theta\, d\theta = \frac{1}{9}\sqrt{3}\int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{9}\sqrt{3}\,(\theta + \tfrac{1}{2}\sin 2\theta) + C$$

$$= \frac{1}{9}\sqrt{3}\ \text{arc tg}\ \frac{2z+1}{3(z-1)}$$

$$+ \frac{2z+1}{12\,(z^2 + z + 1)} + C$$

M a k a :

$$\int \frac{6z^2}{(z^3-1)^2}\, dz = -\frac{2}{3(z-1)} + \frac{2}{3}\ln(z-1) - \frac{1}{z^2+z+1}$$

$$-\frac{1}{3}\ln(z^2+z+1) + \frac{2z+1}{12\,(z^2+z+1)} + \frac{1}{9}\sqrt{3}\ \text{arc tg}\ \frac{2z+1}{3}$$

$$-\frac{10}{9}\sqrt{3}\ \text{arc tg}\ \frac{2z+1}{3} + C.$$

$$= -\frac{2}{3(z-1)} + \frac{1}{3}\ln\frac{(z-1)^2}{z^2+z+1} + \frac{2z-11}{12\,(z^2+z+1)}$$

$$-\sqrt{3}\ \text{arc tg}\ \frac{2z+1}{3} + C.$$

Jadi, hasil terakhir dari integral , adalah :

$$\int \sqrt[3]{\frac{x+1}{x-1}}\, dx = \frac{2\sqrt[3]{x-1}}{3\,(\sqrt[3]{x+1} - \sqrt[3]{x-1})}$$

$$-\frac{1}{3}\ln\frac{(\sqrt[3]{x+1} - \sqrt[3]{(x-1)^2}}{\sqrt[3]{(x+1)^2} + \sqrt[3]{(x^2-1)} + \sqrt[3]{(x-1)^2}}$$

$$- \frac{(2\sqrt[3]{x+1} - 11\sqrt[3]{(x-1)}\sqrt[3]{(x-1)}}{12\left(\sqrt[3]{(x+1)^2} + \sqrt[3]{(x^2-1)} + \sqrt[3]{(x-1)^2}\right.}$$

$$+ \sqrt{3}\ \text{arc tg}\ \frac{2\sqrt[3]{(x+1)} + \sqrt[3]{(x-1)}}{\left(\sqrt[3]{x-1}\right)\sqrt{3}} + C.$$

(ii) Integral dari bentuk :

$$\int \frac{dx}{(hx+k)^n \sqrt{ax^2 + bx + c}}$$

Untuk menyelesaikan bentuk integral tersebut, diambil substitusi :

$$hk + k = \frac{1}{U}\ .$$

Contoh - contoh soal :

3). Tentukan : $\int \frac{dx}{(3x+2)\sqrt{7x^2 + 6x + 1}}$

Penyelesaian :

Substitusi : $3x + 2 = \frac{1}{U}$, $x = \frac{1 - 2U}{3U}$ dan $dx = -\frac{1}{3U^2}\,dU$

Sedang : $7x^2 + 6x + 1 = 7\left(\frac{1-2U}{3U}\right)^2 + 6\left(\frac{1-2U}{3U}\right) + 1$

$$= \frac{U^2 - 10U + 7}{9U^2}$$

M a k a :

$$\int \frac{dx}{(3x+2)\sqrt{7x^2 + 6x + 1}} = \int \frac{-dU / 3U^2}{\frac{1}{U}\sqrt{\frac{U^2 - 10U + 7}{9U^2}}}$$

$$= -\int \frac{dU}{\sqrt{U^2 - 10U + 7}} = -\int \frac{dU}{\sqrt{(U-5)^2 - 18}}$$

$$= -\ln\left| U - 5 + \sqrt{U^2 - 10U + 7} \right| + C$$

$$= \ln \left| \frac{3x + 2}{5x + 3 - \sqrt{7x^2 + 6x + 1}} \right| + C.$$

4) Tentukan : $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 1}}$

Penyelesaian :

Andaikan : $x - 2 = \frac{1}{U}$, $x = \frac{2U + 1}{U}$ dan $dx = -\frac{1}{U^2} dU$

Sedang : $x^2 - 4x + 1 = (x - 2)^2 - 3 = \frac{1 - 3U^2}{U^2}$

M a k a :

$$\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 1}} = \int \frac{-dU / 2}{\frac{1}{U}\sqrt{\frac{1 - 3U^2}{U^2}}}$$

$$= \int \frac{-dU}{1 - 3U^2} = -\frac{1}{3}\sqrt{3} \text{ arc sin } U\ 3 + C$$

$$= -\frac{1}{3}\sqrt{3} \text{ arc sin } \frac{\sqrt{3}}{x - 2} + C$$

5) Tentukan : $\int \frac{dx}{(x + 1)^3 \sqrt{x^2 + 2x}}$

Penyelesaian :

Andaikan : $x + 1 = \frac{1}{U}$, $x = \frac{1 - 4}{U}$ dan $dx = -\frac{1}{U^2} dU$

Sedang : $x^2 + 2x = (x + 1)^2 - 1 = \frac{1}{U^2} - 1$

$$= \frac{1 - U^2}{U^2}$$

M a k a :

$$\int \frac{dx}{(x + 1)^3 \sqrt{x^2 + 2x}} = -\int \frac{U^2}{\sqrt{1 - U^2}} dU = -\int \left(\sqrt{1 - U^2} - \frac{1}{\sqrt{1 - U^2}} \right) dU$$

$$= \tfrac{1}{2}\, U \sqrt{1 - U^2} - \tfrac{1}{2} \arcsin U + C.$$

J a d i :

$$\int \frac{dx}{(x+1)^3 \sqrt{x^2 + 2x}} = \frac{\sqrt{x^2 + 2x}}{2 (x+1)^2} - \tfrac{1}{2} \arcsin \frac{1}{x+1} + C.$$

(iii) Integral dari bentuk , yang memuat :

$$\sqrt{p + q\,x + x^2}$$

Untuk menyelesaikan bentuk integral diatas, diambil Substitusi :

$$p + qx + x^2 = (U - x)^2$$

Contoh - contoh soal

6). Tentukan : $\int \frac{dx}{x \sqrt{x^2 + 2x + 2}}$

Penyelesaian :

Andaikan : $x^2 + 2x + 2 = (U - x)^2 = U^2 - 2xU + x^2$

$$x = \frac{U^2 - 2}{2 (1 + U)}, \quad dx = \tfrac{1}{2} \cdot \frac{U^2 + 2U + 2}{(U+1)^2}\, dU$$

Sedang : $\sqrt{x^2 + 2x + 2} = \tfrac{1}{2} \cdot \frac{U^2 + 2U + 2}{U + 1}$

Sekarang :

$$\int \frac{dx}{x \sqrt{x^2 + 2x + 2}} = \int \frac{\frac{U^2 + 2U + 2}{2 (U+1)^2}\, dU}{\frac{U^2 - 2}{2 (U+1)} \cdot \tfrac{1}{2} \cdot \frac{U^2 + 2U + 2}{U + 1}}$$

$$= \int \frac{2\, dU}{U^2 - 2} = \frac{1}{2} \ln \left| \frac{U - \sqrt{2}}{U + \sqrt{2}} \right| + C.$$

$$= \tfrac{1}{2} \sqrt{2} \ln \left| \frac{x - \sqrt{2} + \sqrt{x^2 + 2x + 2}}{x + \sqrt{2} + \sqrt{x^2 + 2x + 2}} \right| + C.$$

7). Tentukan : $\int \frac{dx}{x\sqrt{x^2 + x + 2}}$

Penyelesaian :

Andaikan : $x^2 + x + 2 = (U - x)^2 = U^2 - 2xU + x^2$

$$x = \frac{U^2 - 2}{1 + 2U} \quad , \quad dx = \frac{2(U^2 + U + 2)}{(1 + 2U)^2} dU$$

Sedang : $\sqrt{x^2 + x + 2} = \frac{U^2 + U + 2}{1 + 2U}$

Sekarang :

$$\int \frac{dx}{x\sqrt{x^2 + x + 2}} = \int \frac{\frac{2(U^2 + U + 2)}{(1 + 2U)^2} dU}{\frac{U^2 - 2}{1 + 2U} \cdot \frac{U^2 + U + 2}{1 + 2U}}$$

$$= 2\int \frac{dU}{U^2 - 2}$$

$$= \frac{1}{2} \ln \left| \frac{U - 2}{U + 2} \right| + C$$

$$= \tfrac{1}{2}\sqrt{2} \ln \left| \frac{\sqrt{x^2 + x + 2} + x - \sqrt{2}}{\sqrt{x^2 + x + 2} + x + \sqrt{2}} \right| + C$$

(iv) Integral dari bentuk , yang memuat :

$$\sqrt{p + q\,p - x^2}$$

dimana bentuk tersebut dapat ditulis :

$$\sqrt{p + qx - x^2} = \sqrt{(x - a)(b - x)}$$

Untuk menyelesaikan integral tersebut diatas, diambil substitusi :

$$p + q\,x - x^2 = (x - a)^2 U^2$$

atau

$$p + q\,x - x^2 = (b - x)^2\,U^2.$$

Contoh - contoh soal :

8). Tentukan : $\int \frac{dx}{x\sqrt{2 - x - x^2}}$

Penyelesaian :

Pandang dulu bentuk : $2 - x - x^2 = (x + 2)(1 - x)$.

Substitusi : $2 - x - x^2 = (x + 2)^2\,U^2$, $x = \frac{1 - 2U^2}{U^2 + 1}$

dan $dx\ \frac{-6U\,dU}{(U^2 + 1)^2}$ sedang, $\sqrt{2 - x - x^2} = \frac{3U}{U^2 + 1}$

J a d i :

$$\int \frac{dx}{x\sqrt{2 - x - x^2}} = \int \frac{\frac{-6U}{(U^2 + 1)^2}}{\frac{1 - 2U^2}{U^2 + 1} \cdot \frac{3U}{U^2 + 1}}\,dU = -2\int \frac{dU}{1 - 2U^2}$$

$$= \tfrac{1}{2}\sqrt{2}\,\ln\left|\frac{U\sqrt{2} - 1}{U\sqrt{2} + 1}\right| + C = \tfrac{1}{2}\sqrt{2}\,\ln\left|\frac{\sqrt{2}\sqrt{\frac{1 - x}{2 + x}} - 1}{\sqrt{2}\sqrt{\frac{1 - x}{2 + x}} + 1}\right| + C$$

$$= \tfrac{1}{2}\sqrt{2}\,\ln\left|\frac{\sqrt{2 - 2x} - \sqrt{2 + x}}{\sqrt{2 - 2x} + \sqrt{2 + x}}\right| + C.$$

9). Tentukan : $\int \frac{2x\,dx}{\sqrt{5x - 6 - x}}$

Penyelesaian :

Pandang dulu bentuk : $5x - 6 - x^2 = (x - 2)(3 - x)$

Substitusi : $5x - 6 - x^2 = (x - 2)^2\,U^2$, $x = \frac{2U^2 + 3}{U^2 + 1}$

dan $dx = -\frac{2U\,dU}{(U^2 + 1)^2}$; $\sqrt{5x - 6 - x^2} = \frac{U}{U^2 + 1}$.

J a d i :

$$\int \frac{2x\,dx}{\sqrt{5x - 6 - x^2}} = \int \frac{2 \cdot \frac{2U^2 + 3}{U^2 + 1} \cdot -\frac{2U\,dU}{(U^2 + 1)^2}}{\frac{U}{U^2 + 1}} =$$

$$= -4\int \frac{2U^2 + 3}{(U^2 + 1)^2}\, dU$$

$$= -4\int \frac{2U^2 + 2}{(U^2 + 1)^2}\, dU - 4\int \frac{dU}{(U^2 + 1)^2}$$

$$= -8\int \frac{dU}{U^2 - 1} - 4\int \frac{dU}{(U^2 + 1)^2}$$

Pandang dulu : $\frac{dU}{(U^2 + 1)^2}$; substitusi : $U = tg\ \theta$

$dU = \sec^2 \theta\, d\theta$

Sedang $U^2 + 1 = tg^2\ \theta + 1 = \sec^2 \theta$

M a k a :

$$\int \frac{dU}{(U^2 + 1)^2} = \int \frac{\sec^2 \theta}{\sec^4 \theta}\, d\theta = \int \cos^2 \theta\, d\theta = \tfrac{1}{2}\int (1 + \cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\ arc\ tg\ U + \frac{U}{2\,(U^2 + 1)^2} + C$$

Hasil terakhir :

$$\int \frac{2x\ dx}{5x - 6x^2} = -8\ arc\ tg\ U - 2\ arc\ tg\ U - \frac{2U}{U^2 + 1} + C.$$

$$= -10\ arc\ tg\ U - \frac{2U}{U^2 + 1} + C$$

$$= -10\ arc\ tg\ \left(\frac{3 - x}{x - 2}\right)^{\frac{1}{2}} - 2\sqrt{5x - 6 - x^2} + C.$$

e. Fungsi rasional dari sin x dan cos x

Jika integran suatu integral , merupakan suatu fungsi rasional dari sin x dan cos x , maka untuk menyelesaikan integral tersebut diambil substitusi :

$$U = tg\ \frac{x}{2} \quad \text{dan} \quad x = 2\ arc\ tg\ U\ ,\ dx = \frac{2\ dU}{1 + U^2}$$

Selanjutnya : $U^2 = tg^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$ dimana didapat ,

$$\cos x = \frac{1 - U^2}{1 + U^2} \quad \text{dan} \quad \sin x = \frac{2U}{1 + U^2}$$

Contoh - contoh soal :

1). Tentukan : $\int \frac{dx}{3 + 4 \cos x}$.

Penyelesaian :

Andaikan : $U = tg \frac{x}{2}$, $x = 2 \text{ arc tg } U$ dan $dx = \frac{2\,dU}{1 + U^2}$

dan $\cos x = \frac{1 - U^2}{1 + U^2}$.

J a d i :

$$\int \frac{dx}{3 + 4 \cos x} = \int \frac{\frac{2\,dU}{1 + U^2}}{3 + 4 \left(\frac{1 - U^2}{1 - U^2}\right)} = 2 \int \frac{dU}{7 - U^2}$$

$$= \frac{1}{\sqrt{7}} \ln \left| \frac{U + \sqrt{7}}{U - \sqrt{7}} \right| + C = \frac{1}{\sqrt{7}} \ln \left| \frac{tg \frac{x}{2} + \sqrt{7}}{tg \frac{x}{2} - \sqrt{7}} \right| + C.$$

2). Tentukan : $\int \frac{dx}{5 + 4 \sin x}$

Penyelesaian :

Andaikan : $x = 2 \text{ arc tg } U$ dan $dx = \frac{2\,dU}{1 + U^2}$

Sedang $\sin x = \frac{2U}{1 + U^2}$

J a d i :

$$= \int \frac{dx}{5 + 4 \sin x} = \int \frac{\frac{2\,dU}{1 + U^2}}{5 + 4 \left(\frac{2U}{1 + U^2}\right)} = 2 \int \frac{dU}{5U^2 + 8U + 5}$$

$$= \frac{2}{5}\int \frac{dU}{(U + \frac{4}{5})^2 + \frac{9}{25}} = \frac{2}{5} \text{ arc tg } \frac{5\,U + 4}{3} + C.$$

$$= \frac{2}{3} \text{ arc tg } \frac{5 \text{ tg } \frac{x}{2} + 4}{3} + C.$$

3). Tentukan : $\int \frac{dx}{2 + \cos x}$

Penyelesaian :

Andaikan : $x = 2 \text{ arc tg } U$ dan $dx = \frac{2\,dU}{1 + U^2}$

Sedang $\cos x$: $\frac{1 - U^2}{1 + U^2}$

J a d i :

$$\int \frac{dx}{2 + \cos x} = \int \frac{\frac{2\,dU}{1 + U^2}}{2 + (\frac{1 - U^2}{1 + U^2})} = 2\int \frac{dU}{3 + U^2} = \frac{2}{\sqrt{3}} \text{ arc tg } \frac{U}{\sqrt{3}} + C.$$

$$= \frac{2}{3}\sqrt{3} \text{ arc tg } (\frac{\sqrt{3}}{3} \text{ tg } \tfrac{1}{2} x) + C.$$

4). Tentukan : $\int \frac{dx}{1 + \sin x - \cos x}$

Penyelesaian :

Andaikan : $x = 2 \text{ arc tg } U$ dan $dx = \frac{2\,dU}{1 + U^2}$

Sedang : $\sin x = \frac{2U}{1 + U^2}$ dan $\cos x = \frac{1 - U^2}{1 + U^2}$

J a d i :

$$\int \frac{dx}{1 + \sin x - \cos x} = \int \frac{\frac{2\,dU}{1 + U^2}}{1 + \frac{2U}{1 + U^2} - \frac{1 - U^2}{1 + U^2}} = \int \frac{dU}{U\,(1 + U)}$$

$$= \int \frac{dU}{U} - \int \frac{dU}{1 + U} = \ln U - \ln (1 + U) + C$$

$$= \ln \frac{U}{1 + U} + C = \ln \left| \frac{\text{tg } \frac{1}{2} x}{1 + \text{tg } \frac{1}{2} x} \right| + C.$$

5) **Tentukan** : $\int \frac{2\ dx}{2 \sin x - \cos x + 3}$

Penyelesaian :

Andaikan : $x = 2 \text{ arc tg } U$ dan $dx = \frac{2\ dU}{1 + U^2}$

Sedang $\sin x = \frac{2U}{1 + U^2}$ dan $\cos x = \frac{1 - U^2}{1 + U^2}$

J a d i :

$$\int \frac{2\ dx}{2 \sin x - \cos x + 3} = \int \frac{2 \cdot \frac{2\ dU}{1 + U^2}}{2 \left(\frac{2U}{1 + U^2}\right) - \left(\frac{1 - U^2}{1 + U^2}\right) + 3}$$

$$= \int \frac{2\ dU}{4U^2 + 4U + 2}$$

$$= \int \frac{dU}{2 (U + \frac{1}{2})^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \text{arc tg} \left| \frac{(U + \frac{1}{2})\sqrt{2}}{\frac{1}{\sqrt{2}}} \right| + C.$$

$$= \text{arc tg } (2U + 1) + C$$

$$= \text{arc tg } \left(2 + 3 \text{ tg } \frac{x}{2}\right) + C.$$

Soal - soal

266. $\int \frac{dx}{1 + \sqrt{x}}$

267. $\int \frac{x\ dx}{(1 + 2x)^{3/2}}$

268. $\int x (1 + x)^{2/3}\ dx$

269. $\int \frac{dx}{x^4 \sqrt{1 + x^2}}$

270. $\int \frac{dx}{(3x + 1)\sqrt{5x^2 + 4x + 1}}$

271. $\int x^8 \sqrt{1 + x^3}\, dx$

272. $\int \frac{x\, dx}{(3 + 5x)^{3/5}}$

273. $\int \frac{dx}{4 + \sqrt{4 + x}}$

274. $\int \frac{\sqrt{a^2 - x^2}}{x}\, dx$

275. $\int \sqrt{4 + x}\, dx$

276. $\int \frac{x^3}{(x^2 + 4)^2}\, dx$

277. $\int \frac{x^5\, dx}{(x^2 - a^2)^3}$

278. $\int \frac{dx}{\sqrt{x + 3}}$

279. $\int \frac{dx}{x\sqrt{4x - x^2}}$

280. $\int x^2 (a + x)^{1/4}\, dx$

281. $\int x^2 \sqrt{1 + x^3}\, dx$

282. $\int e^{2x} \sqrt{e^x + 1}\, dx$

283. $\int \frac{dx}{x^2 \sqrt{x^2 - a^2}}$

284. $\int \frac{dx}{(2 + 3x^2)^{3/2}}$

285. $\int \frac{dx}{\sqrt{1 + e^{2x}}}$

286. $\int \frac{x^2\, dx}{(a^3 - x^3)^{3/4}}$

287. $\int \frac{\sqrt{x - a}}{x}\, dx$

288. $\int \frac{dx}{x^3 (a^4 + x^4)}$

289. $\int \frac{\sqrt{a - x}}{\sqrt{x}}\, dx$

290. $\int \cos \sqrt{x}\, dx$

291. $\int \frac{dx}{(x^2 + 9)^{3/2}}$

292. $\int \frac{x\, dx}{\sqrt{16 + x^2}}$

293. $\int \frac{(x^2 - a^2)^{5/2}}{x}\, dx$

294. $\int (1 + \sqrt{x})^{3/2}\, dx$

295. $\int \frac{e^x}{\sqrt{4 - e^{2x}}}\, dx$

296. $\int \frac{x^2\, dx}{(1 + x)^{1/3}}\, dx$

297. $\int \frac{e^{2x}\, dx}{\sqrt{e^x - 4}}$

298. $\int \frac{dx}{x\sqrt{4x + x^2}}$

299. $\int x\sqrt{x^2 - a^2}\, dx$

300. $\int \sqrt{e^x + 9}\, dx$

301. $\int x^3 \sqrt{a^2 - x^2}\, dx$

302. $\int \frac{dx}{(9x^2 + 25)^{3/2}}$

303. $\int \frac{dx}{(x^2 + 4x + 8)^2}$

304. $\int \frac{1}{1 + e^{2x}}\, dx$

305. $\int \frac{x^2 - a^2}{x}\, dx$

306. $\int \frac{(x^2 - 7)^{3/2}}{x}\, dx$

307. $\int \frac{dx}{\sqrt{1 + \sqrt{x}}}$

308. $\int \frac{\sqrt{x}\, dx}{(1 + 2x)^2}$

309. $\int x \cos^2 x\, dx$

310. $\int \frac{(x^2 - 7)^{3/2}}{x}\, dx$

311. $\int \frac{dx}{x^3 + 3x}$

312. $\int \frac{x^3}{(x^2 + 4)^4}\, dx$

313. $\int \frac{\operatorname{cosec}^2 \theta \operatorname{ctg} \theta}{1 - \operatorname{ctg}^2 \theta}\, dx$

314. $\int \frac{\sin x \cos^2 x\, dx}{1 - 2 \cos x}$

315. $\int \frac{dx}{x^2 (x + a)}$

316. $\int \frac{x\, dx}{x^2 - 6x - 7}$

317. $\int \frac{dx}{x^4 - 2x^3 - 5x^2 + 6x}$

318. $\int \frac{6x^2 + 10x + 2}{x^3 + 3x^2 + 2x}\, dx$

319. $\int \frac{x\, dx}{x^4 + 3x^2 - 4}$

320. $\int \frac{x^4 + 3x^3 - 5x^2 - 4x + 17}{x^3 + x^2 - 5x + 3}\, dx$

321. $\int \frac{dx}{x^3 + x^4}$

322. $\int \frac{dx}{x^3 + x^2 - 4x - 4}$

323. $\int \frac{dx}{x^2 (x^2 - 2x + 1)}$

324. $\int \frac{dx}{x (a^4 + x^4)}$

325. $\int \frac{dx}{(x^2 - 4)^2}$

326. $\int \frac{x\, dx}{(x + 1)^2 (x - 2)}$

327. $\int \frac{\operatorname{tg} x\, dx}{x}$

328. $\int \frac{dx}{x^2 (x^2 + 4)}$

329. $\int \frac{x^4\, dx}{x^2 + 4}$

330. $\int \frac{\sec^2 \theta \operatorname{tg}^3 \theta}{\sqrt{9 + \operatorname{tg}^2 \theta}}\, d\theta$

331. $\int \sec^2 x\, dx$

332. $\int \frac{dx}{x^2 - 3}$.

333. $\int \frac{dx}{(x^2 + 6x + 34)^2}$

334. $\int \frac{x^3\,dx}{(2 - x^2)^{5/2}}$

335. $\int \frac{dx}{x(9 + 4x^2)}$

336. $\int \frac{dx}{x^2 + 2x - 3}$

337. $\int \frac{x^4 + x^3 - 8x^2 - 5x + 6}{x^3 + x - 10x + 8}\,dx$

338. $\int \frac{dx}{x(x^2 - 4)}$

339. $\int \frac{x^3\,dx}{x^2 - 4x + 3}$

340. $\int \frac{x^2\,dx}{x^6 - 5x^3 + 6}$

341. $\int \frac{e^x\,dx}{e^{2x} + e^x - 2}$

342. $\int \frac{dx}{x(x - 1)^2}$

343. $\int \frac{\sqrt{x}\,dx}{(1 + 2x)^2}$

344. $\int \frac{dx}{\sqrt{a^x - 1}}$

345. $\int \frac{dx}{x^3 - 5x^2 + 17 + 13}$

346. $\int \frac{2x^4 - 2x + 1}{x^4(2x - 1)}$

347 $\int \frac{x^4 + x^2 + 2x + 4}{x^3 + 4x}$

348. $\int \frac{x^4 + x^3 + 16}{x(x^2 + 4)^2}\,dx$

349. $\int \frac{2y^3 + y^2 + 2y + 2}{y^4 + 3y^2 + 2}\,dy$

350. $\int \frac{3x^2 - 13x - 59}{(x + 2)^2(x - 5)}\,dx$

351. $\int \frac{x^2 - 2x + 2}{x^3 + 2x^2 - 5x - 6}\,dx$

352. $\int \frac{2x^2 + 3x + 2}{(x + 2)(x^2 + 2x + 2)}$

353. $\int \frac{dx}{x^3 + 3x^2 + 3x + 2}$

354. $\int \frac{x^5 + 9x^3 - 9x^2 - 9}{x^3 + 9x}$

355. $\int \frac{3x^3 + 3x + 1}{x^4 + 3x^2}\,dx$

356. $\int \frac{(2x + 1)\,dx}{x^3 - 5x^2 + 6x}$

357. $\int \frac{4x^2 + 2x + 8}{x(x^2 + 2)^2}\,dx$

358. $\int \frac{dx}{x^3 + x^2 + x}$

359. $\int \frac{4x^2 + 6}{x^3 + 3x}\,dx$

360. $\int \sqrt{e^x - 25}\,dx$

361. $\int \frac{dx}{1 - e^x}$

362. $\int \frac{2x^5 + 3x^4 - x^3 + 6x^2 - 3x + 2}{x^4 + 2x^3}\,dx$

363. $\int \frac{3x^2 - 22x + 19}{(x + 2)(x - 3)^2}\,dx$

364. $\int \frac{7 + 9x + 9x^2 + x^3 - x^4}{(x + 2)(x^2 + 4x + 13)^2}\,dx$

365. $\int \frac{x^4 - 5x^3 - 30x^2 - 73x - 1}{(x - 1)(x^2 + 2x + 2)^2}\,dx$

366. $\int \frac{6x^3 + 2x^2 + 18x - 1}{x^4 + 5x^2 + 4}\,dx$

367. $\int \frac{2x^3 + 5x^2 + 26x + 15}{x^4 + 2x^3 + 4x^2 - 2x - 5}\,dx$

368. $\int \frac{x^4 + 3}{(x + 1)(x^2 + 1)}\,dx$

369. $\int \frac{3x^3 + x^2 + 3}{x^4 + 3x^2}\,dx$

370. $\int \left(\frac{t + 3}{t^2 + 4t + 5}\right)^2 dt$

371. $\int \frac{5x^2 + 12x + 9}{x^3 + 3x^2 + 3x}\,dx$

372. $\int \frac{6x^2 + 3x + 4}{x^3 + 2x}\,dx$

373. $\int \frac{x^2 + x - 10}{(2x - 3)(x^2 + 4)}\,dx$

374. $\int \frac{2x^3 - 3x}{x^4 - 3x^2 + 1}\,dx$

375. $\int \frac{dx}{2x^2 - 3x - 2}$

376. $\int \frac{x\,dx}{(x - 1)(x^2 + 4)}$

377. $\int x^3 \sqrt{16 - x^2}\,dx$

378. $\int \frac{dx}{e^x (e^x + 1)^2}$

379. $\int \frac{dx}{x^4 + 9x^2}$

380. $\int \frac{dx}{x^2 (x^2 + 1)^2}$

381. $\int \frac{dx}{x (x^2 + 1)^2}$

382. $\int \frac{dx}{x^3 + 1}$

383. $\int \frac{dx}{x^3 + a^3}$

384. $\int \frac{x^2 + x}{(x - 1)(x^2 + 1)}\,dx$

385. $\int \frac{dx}{(x^2 + 2x + 2)(x^2 + 2x + 5)}$

386. $\int \frac{dx}{(x^2 - 4x + 3)(x^2 + 4x + 5)}$

387. $\int \frac{x^5\,dx}{x^4 + x^3 + 2x^2 + x + 1}$

388. $\int \frac{x^5\,dx}{(x^3 + 1)(x^3 + 8)}$

389. $\int \frac{x^4 + 8x^3 - x^2 + 2x + 1}{(x^2 + x)(x^3 + 1)}\,dx$

390. $\int \frac{dx}{(2x - 1)\sqrt{6x^2 - 12x + 3}}$

391. $\int \frac{dx}{\sqrt{x^2 + x + 1}}$

392. $\int \frac{dx}{x\sqrt{2 + x - x^2}}$

393. $\int \frac{dx}{x\sqrt{x^2 + x + 1}}$

394. $\int \frac{\sqrt{x^2 + 2x}}{x^2}\,dx$

395. $\int \frac{e^x\,dx}{(e^x - 1)^2 (e^{2x} - 1)}$

396. $\int \frac{x\,dx}{x^6 - 8}$

397. $\int \frac{dx}{x^3 + 2x^2 + 2x}$

398. $\int \frac{dx}{x^4 + 4}$

399. $\int \frac{dx}{x^5 + 4x^3}$

400. $\int \frac{t^5\,dt}{(t^2 + 4)^2}$

401. $\int \frac{x - 18}{4x^3 + 9x}\,dx$

402. $\int \frac{x^7 + x^3}{x^{12} - 2x^4 + 1}\,dx$

403. $\int \frac{x^3 + x^2 - 5x + 15}{(x^2 + 5)(x^2 + 2x + 3}\,dx$

404. $\int \frac{x^6 + 7x^5 + 15x^4 + 32x^3 + 23x^2 + 25x - 3}{(x^2 + x + 2)^2 (x^2 + 1)}\,dx$

405. $\int \frac{dx}{x^4 (x^3 + 1)^2}$

406. $\int \frac{dx}{e^{2x} - 3e^x}$

407. $\int \frac{dx}{x\sqrt{3x - 2 - x^2}}$

408. $\int \frac{dx}{x\sqrt{x^2 - 2 - x^2}}$

409. $\int \frac{dx}{x\sqrt{x^2 + x + 4}}$

410. $\int \frac{dx}{x\sqrt{x^2 + x + 4}}$

411. $\int \frac{\sqrt{x^2 + 4x}}{x^2}\,dx$

412. $\int \frac{dx}{x\sqrt{x^2 - 2a}}$

413. $\int \frac{e^x\,dx}{\sqrt{1 - e^{2x}}}$

414. $\int \frac{dx}{\sqrt{2 + x - x^2}}$

415. $\int \frac{5x + 9}{(x - 9)\, x^{3/2}}\, dx$

416. $\int \frac{x^2\, dx}{(4x + 1)^{5/2}}$

417. $\int \frac{dx}{(x - 2)^{1/2} - (x - 2)^{3/4}}$

418. $\int \frac{x + 2}{x\sqrt{x - 3}}\, dx$

419. $\int x^5 \sqrt{1 + x^3}\, dx$

420. $\int \frac{dx}{x^2\, (1 + x^3)^{2/3}}$

421. $\int \frac{dx}{x^2\, (1 + x^4)^{3/4}}$

422. $\int \frac{x^5 + 2x}{(1 + x^3)^{3/2}}\, dx$

423. $\int \frac{2\sqrt{1 + x^4}}{x^2}\, dx$

424. $\int \frac{x^5\, dx}{(a + bx^3)^{3/2}}$

425. $\int \frac{dx}{x^3\, (1 + x^3)^{3/2}}$

426. $\int \frac{dx}{(1 + x)^{3/2}}$

427. $\int \frac{dx}{(6x - x^2)^{3/2}}$

428. $\int \left(\sqrt{\frac{a + x}{a - x}} - \sqrt{\frac{a - x}{a + x}} \right) dx$

429. $\int \frac{dx}{x\sqrt{x^2 + 3x + 25}}$

430. $\int \frac{(2x + 1)\, dx}{\sqrt{x^2 + x + 1}}$

431. $\int \frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} - 1}\, dx$

432. $\int \frac{dx}{x - x^{4/3}}\, dx$

433. $\int \frac{dx}{x + 2\sqrt{x + 5}}$

434. $\int \frac{dx}{(x + 1)^{1/4} - (x + 1)^{5/4}}$

435. $\int \frac{2 - \sqrt{2 + 3}}{1 - 2x}\, dx$

436. $\int x^5\, (8 + x^3)^{3/2}\, dx$

437. $\int \frac{y\, dy}{(2y + 3)^{4/3}}$

438. $\int \frac{(x + 3)\, dx}{(x + 5)\sqrt{x + 4}}$

439. $\int \frac{x^5\, dx}{\sqrt{1 + x^3}}$

440. $\int \frac{x^{2/3} - x^{1/3}}{6\, x^{\frac{1}{4}}}\, dx$

441. $\int \frac{dx}{x^3\,(1 + x^3)^{1/3}}$

442. $\int \frac{\sqrt{x}\; dx}{x^3 + 2\,x^2 - 3\,x}$

443. $\int \frac{e^{4x}\; dx}{\sqrt{1 + e^{2x}}}$

444. $\int \frac{dx}{x\sqrt{x^2 - x + 2}}$

445. $\int \frac{(x + 5)\; dx}{(x + 4)\sqrt{x + 2}}$

446. $\int \frac{\sqrt{x^2 + 4}}{x^6}\, dx$

447. $\int \frac{\sqrt{16 - u^2}}{u^2}\, du$

448. $\int \frac{dx}{x\sqrt{5\,x - 6 - x^2}}$

449. $\int \frac{dx}{x^2\sqrt{1 + 2\,x + 3\,x^2}}$

450. $\int \frac{dx}{\sqrt{5\,x - 6 - x^2}}$

451. $\int \frac{dx}{x\sqrt{x^2 + 2\,x - 1}}$

452. $\int \frac{x^2 dx}{\sqrt{x^2 - 6}}$

453. $\int \frac{dx}{x\sqrt{x^2 + 4\,x - 4}}$

454. $\int \frac{x^2\, dx}{(9 - x^2)^{3/2}}$

455. $\int \frac{dx}{x\sqrt{25 - x^2}}$

456. $\int \frac{dx}{x^3\sqrt{x^2 + 1}}$

457. $\int \frac{x^2\, dx}{\sqrt{x^2 + 5}}$

458. $\int \frac{dx}{(x^2 + 2)^{3/2}}$

459. $\int \frac{x^2\, dx}{(x^2 + 8)^{3/2}}$

460. $\int \frac{dx}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}$

461. $\int \frac{\sqrt{100 - u^2}}{u}\, du$

462. $\int \frac{dx}{x^3\sqrt{4 - x^2}}$

463. $\int \frac{\sqrt{x^2 + 9}}{x^2}\, dx$

464. $\int \frac{\sqrt{y^2 - 9}}{y}\, dy$

465. $\int \frac{dx}{x\sqrt{1 + 4x + 5x^2}}$

466. $\int \frac{3\sqrt[3]{x} + 2\sqrt[6]{x}}{(x + \sqrt[3]{x - 2})^2}\, dx$

467. $\int \frac{4\, dx}{x\sqrt{x^2 - 2x + 3}}$

468. $\int \frac{dx}{x\sqrt{3x^2 - 2x - 1}}$

469. $\int \frac{x^2\, dx}{\sqrt{4 - x^2}}$

470. $\int \frac{dx}{x^2\sqrt{27x^2 + 6x - 1}}$

471. $\int \frac{4x\, dx}{(x^2 - 2x + 3)^{3/2}}$

472. $\int \frac{dx}{x^2\sqrt{25 - x^2}}$

473. $\int \frac{dy}{y^2\sqrt{y^2 - 7}}$

474. $\int \frac{dx}{x^3\sqrt{x^2 - 9}}$

475. $\int \frac{dx}{(5 - x^2)^{3/2}}$

476. $\int \frac{dx}{x\sqrt{x^2 + 4}}$

477. $\int \frac{dx}{x + 1 + \sqrt{(x + 1)^3}}$

478. $\int \frac{\sqrt{x^2 + 16}}{x}\, dx$

479. $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$

480. $\int \frac{dx}{x^{5/8} - x^{1/8}}$

481. $\int x(1 + x^3)^{1/3}\, dx$

482. $\int \frac{dU}{(U^2 - 3)^{3/2}}$

483. $\int \frac{dx}{x^4\sqrt{x^2 - 5}}$

484. $\int \frac{x^3}{\sqrt{x - 1}}\, dx$

485. $\int \frac{dx}{x^6\sqrt{1 + x^2}}$

486. $\int \left| \ln (x + \sqrt{1 + x^2}) \right|^2 dx$

487. $\int \frac{dx}{(x - 1)^2\sqrt{6 - 6x + x^2}}$

488. $\int \frac{dx}{(x^2 - 4)\sqrt{x^2 - 1}}$

489. $\int x\sqrt{\frac{x-1}{x+1}}\,dx$

490. $\int \frac{1}{x}\sqrt{\frac{x^{10}-1}{x^{10}+1}}\,dx$

491. $\int \frac{x^3\,dx}{\sqrt{2x-x^2}}$

492. $\int \frac{\sqrt{x}}{x+2}\,dx$

493. $\int \frac{dx}{(2-x)\sqrt{1-x}}$

494. $\int \frac{\sqrt{x-1}}{\sqrt[3]{x+1}}\,dx$

495. $\int \frac{x^2+x+1}{x\sqrt{x^2-x+1}}\,dx$

496. $\int \frac{dx}{\sqrt[4]{1+x^4}}$

497. $\int (\sqrt{x+1})(x-\sqrt{x+1})\,dx$

498. $\int \frac{dx}{(5-x)^{1/4}+(5-x)^{1/2}}$

499. $\int (x^2-1)\,10^{-2x}\,dx$

500. $\int \frac{dx}{(x+1)^3\sqrt{x^2+2x}}$

501. $\int \frac{x^6}{\sqrt{1+x^2}}\,dx$

502. $\int \frac{x^3\,dx}{(1+2x^2)^{3/2}}$

503. $\int \frac{dx}{x^2(2+x^3)^{5/3}}$

504. $\int x(2x+5)^{10}\,dx$

505. $\int \frac{\ln x\,dx}{x\sqrt{1-4\ln x-\ln^2 x}}$

506. $\int \frac{dx}{x\sqrt{x^2-2}}$

507. $\int \sqrt[4]{\frac{1-x}{1+x}}\,dx$

508. $\int x^3\sqrt{\frac{a^2-x^2}{a^2+x^2}}\,dx$

509. $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$

510. $\int \sqrt{x^2\,3x}\,dx$

511. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx$

512. $\int \frac{dx}{x^5\sqrt{x^2-1}}$

513. $\int \frac{x^5\,dx}{\sqrt{1-x^2}}$

514. $\int \frac{dx}{x\sqrt[3]{1+x^5}}$

515. $\int (\sqrt{x} - 1)\ (x - \sqrt{x} - 1)\ dx$

516. $\int \frac{dx}{\sqrt{x^3 + \sqrt[3]{1 + \sqrt[4]{x^3}}}}$

517. $\int \frac{\ln 2x}{\ln 4x}\ dx$

518. $\int \frac{dx}{e^{2x} - 3e^{x}}$

519. $\int \frac{(x^2 + 1)\ (x^2 - 2)}{3x^2}\ dx$

520. $\int (2 + \frac{x}{2x^2 + 1}) \cdot \frac{dx}{2x^2 + 1}$

521. $\int \frac{3\sqrt[3]{1 + \sqrt[4]{x}}}{\sqrt{x}}\ dx$

522. $\int \frac{1 - \sqrt{3x + 2}}{1 + \sqrt{3x + 2}}\ dx$

523. $\int \frac{\cos\theta\ d\theta}{\sin^2\theta\ (1 - \sin\theta)}$

524. $\int \frac{1}{1 \sin\theta}\ \operatorname{tg}\theta\ d\theta$

525. $\int \frac{dx}{4 + 2\cos 3x}$

526. $\int \frac{\sin 6x}{1 - 2\cos 6x}\ dx$

527. $\int \frac{\sin x}{1 + \sin x}\ dx$

528 $\int \frac{dx}{1 + \sin x - \cos x}$

529. $\int \frac{dx}{4 + 5\cos x}$

530. $\int \frac{\sin 3x\ dx}{5 + 4\sin 3x}$

531. $\int x\ \operatorname{tg}^4 x\ dx$

532. $\int \sin^2 x \cos^6 x\ dx$

533. $\int \frac{dx}{2 + \sin x}$

534. $\int \cos 6x \cos 9x\ dx$

535. $\int \frac{dx}{(\sqrt[3]{x^2} + \sqrt[3]{x})^2}$

536. $\int x\ (5x^2 - 3)^7\ dx$

537. $\int \frac{2x - 8}{\sqrt{1 - x - x^2}}\ dx$

538. $\int \frac{\sqrt{2 + x^2} - \sqrt{2 - x^2}}{\sqrt{4 - x^4}}\ dx$

539. $\int \frac{dx}{3\ (1 - x^2) - (5 + 4x)\sqrt{1 - x^2}}$

540. $\int \frac{(\sqrt{a} - \sqrt{2x})^4}{\sqrt{ax}}\ dx$

541. $\int \frac{\ln (x + \sqrt{x^2 + 1})}{1 + x^2}\ dx$

542. $\int \frac{\cos\theta\ d\theta}{(2 - \sin\theta)^2\ (1 - \sin\theta)}$

543. $\int \frac{\operatorname{ctg}\theta}{1-\cos\theta}\,d\theta$

544. $\int \frac{dx}{1+2\cos x}$

545. $\int \frac{\operatorname{tg}\theta}{1-\cos\theta}\,d\theta$

546. $\int \frac{dx}{3-5\sin x}$

547. $\int \frac{dx}{4+\cos x}$

548. $\int \frac{dx}{1+\cos x}$

549. $\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx$

550. $\int \sqrt{1+2\sin\theta}\cdot\cos\theta\sin\theta\,d\theta$

551. $\int x\cos^2 x\,dx$

552. $\int \frac{1+\cos x}{\cos x\,(1-\sin x)}\,dx$

553. $\int \sin^{10} x\cos^3 x\,dx$

554. $\int \frac{dx}{1+\sin x+\cos x}$

555. $\int \frac{dx}{13\cos x-5}$

556. $\int \frac{dx}{\sin x+\operatorname{tg} x}$

557. $\int \frac{dx}{3\sec x-4}$

558. $\int \frac{dx}{1+2\sin x}$

559. $\int \frac{1}{x^2}\sqrt{\operatorname{tg}\frac{1}{x}}\,dx$

560. $\int \frac{dx}{13-5\cos x}$

561. $\int \frac{\sec^4\theta\operatorname{tg}\theta}{\sqrt{\sec^2\theta+1}}\,d\theta$

562. $\int \frac{\sin c\ \cos x}{\sin^4 x+\cos^4 x}\,dx$

563. $\int \frac{\cos 4x+2\cos 2x}{\cos^2 x}\,dx$

564. $\int \frac{dx}{\sqrt{\sin x\cos^3 x}}$

565. $\int \frac{\sin^3 x}{\sqrt[5]{\cos^3 x}}\,dx$

566. $\int \sin\left(\frac{\pi}{4}-x\right)\sin\left(\frac{\pi}{4}+x\right)dx$

567. $\int \frac{dx}{(\sin x+\cos x)^2}$

568. $\int \left(\operatorname{tg}^3\frac{x}{3}+\operatorname{tg}^4\frac{x}{4}\right)dx$

569. $\int \frac{\sin x\,dx}{\sin^3 x+\cos^3 x}$

570. $\int \sin 2x \cos (2x + 6)\, dx$

571. $\int \frac{dx}{\cos^2 x + \sin 2x + 2 \sin^2 x}$

572. $\int \frac{dx}{5 + 4 \cos x}$

573. $\int \frac{dx}{3 + \cos x}$

574. $\int \frac{dx}{4 \sec x + 5}$

575. $\int \frac{dx}{\operatorname{ctg} x - \operatorname{cosec} x}$

576. $\int \frac{\sin x\, dx}{5 + 4 \sin x}$

577. $\int \frac{dx}{2 \sin^2 x - 3 \cos x}$

578. $\int \frac{1 - \sin x + \cos x}{1 + \sin x - \cos x}\, dx$

579. $\int \frac{dx}{\sqrt{1 + \sin^2 x}}$

580. $\int \frac{\sin x\, dx}{\sqrt{\cos^2 x + 4 \cos x + 1}}$

581. $\int \frac{\cos x\, dx}{\sin^2 x - 6 \sin x + 12}$

582. $\int \frac{dx}{\sin^2 x - 5 \sin x \cos x}$

583. $\int \frac{(\arcsin x)^2}{\sqrt{1 - x^2}}\, dx$

584. $\int \sqrt{\operatorname{ctg} x}\, dx$

585. $\int \frac{dx}{3 \cos (5x - \frac{\pi}{4})}$

586. $\int \sin 10x \sin 15x\, dx$

587. $\int \frac{\arcsin (\operatorname{tg} x)}{\sin^2 x}\, dx$

588. $\int \frac{\sin (x + \frac{\pi}{4})}{\sin x \cos x}\, dx$

589. $\int \frac{dx}{\sin^2 x + 3 \sin x \cos x - \cos^2 x}$

□ BAB III
INTEGRASI FUNGSI HIPERBOLIK

1. RUMUS-RUMUS DASAR FUNGSI HIPERBOLIK

Fungsi hiperbolik merupakan fungsi yang dibentuk oleh kombinasi dari eksponensiil e^{x} dan e^{-x}. Oleh karena itu, mengintegral fungsi hiperbolik berarti juga mengintegral fungsi eksponensiil yang telah dikenal cara pengintegralannya.

Kita pelajari fungsi hiperbolik ini, oleh karena fungsi hiperbolik banyak digunakan untuk menyelesaikan soal-soal dalam fisika dan juga digunakan untuk menyelesaikan persamaan differensial.

Dibawah ini ditulis bentuk - bentuk dan rumus - rumus dasar fungsi hiperbolik.

A). Bentuk - bentuk fungsi hiperbolik

1). $\sin U = \dfrac{e^{U} - e^{-U}}{2}$; 2). $\cos U = \dfrac{e^{U} + e^{-U}}{2}$

3). $\operatorname{tg} U = \dfrac{e^{U} - e^{-U}}{e^{U} + e^{-U}}$; 4). $\operatorname{ctg} U = \dfrac{e^{U} + e^{-U}}{e^{U} - e^{-U}}$

5). $\sec U = \dfrac{2}{e^{U} + e^{-U}}$; 6). $\operatorname{cosec} U = \dfrac{2}{e^{U} - e^{-U}}$

B). Rumus - rumus dasar

1). $\cos^{2} U - \sin^{2} U = 1$; 2). $1 + 2\sin^{2} U = \cos 2U$

3). $1 - \operatorname{tg}^{2} U = \sec^{2} U$; 4). $2\cos^{2} U - 1 = \cos 2U$

5). $1 - \text{ctg}^2 U = -\text{cosec}^2 U$; 6). $\cos^2 U + \sin^2 U = \cos 2U$

7). $\sin 2U = 2 \sin U \cos U$

8). $\sin (U \pm V) = \sin U \cos V \pm \cos U \sin V$

9). $\cos (U \pm V) = \cos U \cos V \pm \sin U \sin V$

10). $\sin U + \sin V = 2 \sin \frac{U + V}{2} \cos \frac{U - V}{2}$

11). $\sin U - \sin V = 2 \cos \frac{U + V}{2} \sin \frac{U - V}{2}$

12). $\cos U + \cos V = 2 \cos \frac{U + V}{2} \cos \frac{U - V}{2}$

13). $\cos U - \cos V = 2 \sin \frac{U + V}{2} \sin \frac{U - V}{2}$

Dari rumus-rumus dasar fungsi hiperbolik diatas, dapat diturunkan rumus-rumus dasar integral untuk fungsi hiperbolik.

2. RUMUS-RUMUS DASAR INTEGRAL DARI FUNGSI HIPERBOLIK

1). $\int \sin U \, dU = \cos U + C$

2). $\int \cos U \, dU = \sin U + C$

3). $\int \sec^2 U \, dU = \text{tg } U + C$

4). $\int \cos^2 U \, dU = -\text{ctg } U + C.$

5). $\int \text{cosec } U \, \text{ctg } U \, dU = -\text{cosec } U + C.$

6). $\int \sec U \, \text{tg } U \, dU = -\sec U + C.$

7). $\int \frac{dU}{\sqrt{a^2 + U^2}} = \text{arc sin } \frac{U}{a} + C.$

8). $\int \frac{dU}{\sqrt{U^2 - a^2}} = \text{arc cos } \frac{U}{a} + C$

9). $\int \frac{dU}{a^2 - U^2} = \begin{cases} \frac{1}{a} \text{ arc tg } U + C, & \text{jika } |U| < 1 \\ \frac{1}{a} \text{ arc ctg } U + C, & \text{jika } |U| > 1 \end{cases}$

10). $\int \frac{dU}{U\sqrt{a^2 - U^2}} = \begin{cases} -\frac{1}{a} \text{ arc sec } \left(\frac{|U|}{a}\right) + C. \\ -\frac{1}{a} \text{ arc cos } \left(\frac{a}{|U|}\right) + C. \end{cases}$

11). $\int \frac{dU}{U\sqrt{a^2 + U^2}} = \begin{cases} -\frac{1}{a} \text{ arc cosec } \left(\frac{|U|}{a}\right) + C. \\ -\frac{1}{a} \text{ arc sin } \left(\frac{a}{|U|}\right) + C. \end{cases}$

Contoh - contoh soal :

1). Tentukan : $\int \sin 2x \, dx$

Penyelesaian :

$$\int \sin 2x \, dx = \tfrac{1}{2} \int \sin 2x \, d(2x) = \tfrac{1}{2} \cos 2x + C.$$

2). Tentukan : $\int \text{cosec}^2 (3x + 1) \, dx$

Penyelesaian :

$$\int \text{cosec}^2 (3x + 1) = \frac{1}{3} \int \text{cosec}^2 (3x + 1) \, d(3x + 1)$$

$$= -\frac{1}{3} \text{ cotg } (3x + 1) + C.$$

3). Tentukan : $\int \cos 4x \, dx$

Penyelesaian :

$$\int \cos 4x \, dx = \tfrac{1}{4} \sin 4x + C.$$

4). Tentukan : $\int \text{tg } \frac{x}{3} \, dx$

Penyelesaian :

$$\int \text{tg } \frac{x}{3} \, dx = \int \frac{\sin x/3}{\cos x/3} \, dx = 3 \int \frac{\sin x/3}{\cos x/3} \, d(x/3)$$

$$= 3 \int \frac{1}{\cos x/3} \, d \cos x/3 = 3 \ln \cos \frac{x}{3} - C.$$

5). Tentukan : $\int \frac{\sec \sqrt{x} \text{ tg } \sqrt{x}}{\sqrt{x}} \, dx$

Penyelesaian :

Bentuk $\frac{dx}{\sqrt{x}} = 2 \, d(\sqrt{x})$, maka integral sekarang :

$$\int \frac{\text{sech } \sqrt{x} \ \text{tg } \sqrt{x}}{\sqrt{x}} \, dx = \int \sec \sqrt{x} \ \text{tg } \sqrt{x} \, . \, 2 \, d(\sqrt{x})$$
$$= -2 \sec \sqrt{x} + C.$$

6). Tentukan : $\int x \sin x^2 \, dx$

Penyelesaian :

$$\int x \sin x^2 \, dx = \tfrac{1}{2} \int \sin x^2 \, . \, 2x \, dx = \tfrac{1}{2} \int \sin x^2 \, dx^2$$
$$= \tfrac{1}{2} \cos x^2 + C.$$

7). Tentukan : $\int x \sin x \, dx$

Penyelesaian :

Untuk menyelesaikan soal ini, digunakan integrasi persiil.

$$\int x \sin x \, dx = \int x \, d(\cos x) = x \cos x - \int \cos x \, dx$$
$$= x \cos x \sin x + C.$$

8). Tentukan : $\int \frac{1}{5 z^2 - 1} \, dz$

Penyelesaian :

$$\int \frac{1}{5 z^2 - 1} \, dz = \int \frac{dz}{(z\sqrt{5})^2 - 1} = - \int \frac{dz}{1 - (z\sqrt{5})^2}$$
$$= -\frac{1}{5} \text{ arc tg } z\sqrt{5} + C.$$

9). Tentukan : $\int \frac{x^2}{16 - x^6} \, dx$

Penyelesaian :

$$\int \frac{x^2}{16 - x^6} \, dx = \frac{1}{3} \int \frac{dx^2}{16 - x^6} = \frac{1}{12} \text{ arc tg } \frac{x^3}{4} + C.$$

10). Tentukan : $\int \frac{dy}{\sqrt{16 y^2 - 9}}$

Penyelesaian :

$$\int \frac{dy}{\sqrt{16 y^2 - 9}} = \int \frac{dy}{\sqrt{(4y)^2 - 3^2}} = \tfrac{1}{4} \text{ arc cos } \frac{4y}{3} + C.$$

11). Tentukan : $\int \frac{\text{tg}^2 x \, dx}{\cos x}$

Penyelesaian :

$$\int \frac{tg^2 x\, dx}{\cos x} = \int \frac{\sin^2 x}{\cos^3 x}\, dx = \int \frac{\cos^2 x - 1}{\cos^3 x}\, dx$$

$$= \int \sec x\, dx - \int \sec^3 x\, dx$$

Dicari dulu bentuk :

$$(i).\ \int \sec x\, dx = \int \frac{2}{e^x + e^{-x}}\, dx = 2\int \frac{e^x}{e^{2x} + 1}\, dx$$

$$= \text{arc tg } e^x + C.$$

$$(ii).\ \int \sec^3 x\, dx = \int \sec x \cdot \sec^2 x\, dx = \int \sec x\, dtg\, x$$

$$= \sec x \text{ tg } x + \int \sec x \text{ tg}^2 x\, dx$$

$$= \sec x \text{ tg } x + \int \sec x\, (1 - \sec^2 x)\, dx$$

$$= \text{sech } x \text{ tg } x + \int \sec x\, dx - \int \sec^3 x\, dx$$

$$2\int \sec^3 x\, dx = \sec x \text{ tg } x + \text{arc tg } e^x \qquad \text{(lihat (i))}$$

$$\int \sec^3 x\, dx = \tfrac{1}{2}(\sec x \text{ tg } x + \text{arc tg } e^x) + C.$$

Jadi hasil seluruhnya :

$$\int \frac{tg^2 x\, dx}{\cos x} = \tfrac{1}{2}(\text{arc tg } e^x - \sec x \text{ tg } x) + C.$$

3. SUBSTITUSI FUNGSI HIPERBOLIK

Didepan telah diketahui adanya substitusi fungsi goniometri, untuk menyelesaikan integral dari bentuk :

$$\int F\left(x, \sqrt{ax^2 + bx + c}\right) dx$$

Substitusi lain, yang dapat digunakan untuk menyelesaikan integral tersebut ialah substitusi fungsi hiperbolik.

Untuk bentuk :

a). $\sqrt{a^2 - b^2 x^2}$, diambil substitusi $x = \frac{a}{b} \text{ tg } U$

b). $\sqrt{a^2 + b^2 x^2}$, diambil substitusi $x = \frac{a}{b} \sin U$

c). $\sqrt{b^2 x^2 - a^2}$, diambil substitusi $x = \frac{a}{b} \cos U$

Contoh - contoh soal :

12). Tentukan : $\int \sqrt{3 - 2x - x^2}\, dx$

Penyelesaian :

Pandang dulu bentuk :

$3 - 2x - x^2 = 4 - (x + 1)^2$, substitusi $x + 1 = 2 \operatorname{tg} U$

$dx = 2 \operatorname{sech}^2 U\, dU$; maka :

$$\int \sqrt{3 - 2x - x^2}\, dx = \int \sqrt{4 - (x + 1)^2}\, dx = \int \sqrt{4 - 4 \operatorname{tg}^2 U}\, .2 \sec^2 U\, dU$$

$$= 4 \int \sec^3 U\, dU = 2 \sec U \operatorname{tg} U + \operatorname{arc\,tg}\, e^U + C.$$

Sedang $\operatorname{tg} U = \frac{x + 1}{2}$, $\frac{e^U - e^{-U}}{e^U + e^{-U}} = \frac{x + 1}{2}$

Maka : $e^U = \frac{3 + x}{1 - x}$; jadi hasil integral yang terakhir :

$$\int \sqrt{3 - 2x - x^2}\, dx = \tfrac{1}{2} (x + 1) \sqrt{3 - 2x - x^2} + 2 \operatorname{arc\,tg} \left(\frac{3 + z}{1 - x}\right) + C.$$

13). Tentukan : $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}}$

Penyelesaian :

Untuk bentuk ini, diambil substitusi $x = a \sinh u$

$dx = a \cosh U\, dU$ dan $x^2 + a^2 = a^2 \sin^2 U + a^2$

$= a^2 \cos^2 U.$

Maka : $\int \frac{dx}{x^2 \sqrt{x^2 + a^2}} = \int \frac{a \cos U\, dU}{(a^2 \sin^2 U) \,.\, (a \cos U)}$

$$= \frac{1}{a^2}\int \operatorname{cosec}^2 U \, dU = -\frac{1}{a^2} \operatorname{ctg} U + C.$$

Oleh karena : $\sin U = \frac{x}{a}$ dan $\cos^2 U = 1 + \sin^2 U$

Maka $\cos U = \frac{1}{a}\sqrt{x^2 + a^2}$ dan $\operatorname{ctg} U = \frac{\sqrt{x^2 + a^2}}{x}$

Jadi , hasil integral seluruhnya :

$$\int \frac{dx}{x^2\sqrt{x^2 + a^2}} = -\frac{1}{a^2} \cdot \frac{x^2 + a^2}{x} + C.$$

14). Tentukan : $\int \frac{dx}{(x^2 - a^2)^{3/2}}$

Penyelesaian :

Untuk bentuk ini , diambil substitusi $x = a \cos U$,

$dx = a \sin U \, dU$ dan $x^2 - a^2 = a^2 \cos^2 U - a^2$

$= a^2 \sin^2 U$

M a k a :

$$\int \frac{dx}{(x^2 - a^2)^{3/2}} = \int \frac{a \sin U \, dU}{(a^2 \sin^2 U)^{3/2}} = \frac{1}{a^2}\int \operatorname{cosec}^2 U \, dU$$

$$= -\frac{1}{a^2} \operatorname{ctg} U + C = \frac{x}{a^2\sqrt{x^2 - a^2}} + C.$$

15). Tentukan : $\int \frac{dx}{(1 + x^2)\sqrt{1 - x^2}}$

Penyelesaian :

Digunakan substitusi : $x = \operatorname{tg} U$, $dx = \sec^2 U \, dU$

Maka :

$$\int \frac{dx}{(1 + x^2)\sqrt{1 - x^2}} = \int \frac{\sec^2 U \, dU}{(1 + \operatorname{tg}^2 U) \cdot \sec U}$$

$$= \int \frac{\cos U}{\cosh^2 U + \sin^2 U}\, dU = \int \frac{\cos U\, dU}{(1 + \sin^2 U) + \sin^2 U}$$

$$= \int \frac{d(\sin U)}{1 + 2 \sin^2 dU} = \frac{1}{\sqrt{2}} \text{ arc tg } (\sqrt{2} \sin U) + C.$$

$$= \frac{1}{\sqrt{2}} \text{ arc tg } \frac{x\sqrt{2}}{\sqrt{1 - x^2}} + C$$

Soal - soal :

590. $\int (2 \sin 5x - 3 \cos 5x)\, dx$

591. $\int z^3 \text{ ctg } z^4\, dz$

592. $\int \text{ctg}^4 U\, dU$

593. $\int \text{tg}^3 7s\, ds$

594. $\int \cos x \sin x\, dx$

595. $\int \sin^3 x\, dx$

596. $\int y \cos^2 (y^2 - 9)\, dy$

597. $\int \text{tg}^4 U\, dU$

598. $\int x \cos x\, dx$

599. $\int \sin x \ln \cos^2 x\, dx$

600. $\int \sin^3 x \cos^2 x\, dx$

601. $\int x \sin x\, dx$

602. $\int \sin x \cos x\, dx$

603. $\int x^2 \cos x\, dx$

604. $\int x^2 \sin x\, dx$

605. $\int \text{cosec}^2 (1 + 3x)\, dx$

606. $\int \frac{dz}{\sin z \text{ tg } z}$

607. $\int \frac{dx}{\sin^2 x + \cos^2 x}$

608. $\int \frac{dx}{2 \sin x + 3 \cos x}$

609. $\int \frac{dx}{\text{tg } x - 1}$

610. $\int \frac{dx}{\sin z \text{ tg } z}$

Gunakan substitusi fungsi hiperbolik:

611. $\int \frac{dx}{4x^2 + 12x + 15}$

612. $\int \sin \frac{y}{3}\, dy$

613. $\int \sec 2x \operatorname{tg} 2x\, dx$

614. $\int \operatorname{ctg}^2 3x\, dx$

615. $\int e^x \cos e^x\, dx$

616. $\int \sin^3 x \cos x\, dx$

617. $\int \cos^4 x\, dx$

618. $\int x^3 \sin x^4\, dx$

619. $\int \sin^4 2u\, du$

620. $\int e^x \sin x\, dx$

621. $\int \sin^2 x \cos^2 x\, dx$

622. $\int \cos^5 x \sin^2 x\, dx$

623. $\int \sin^4 x\, dx$

624. $\int \operatorname{cosec}^3 x\, dx$

625. $\int e^{2x} \cos x\, dx$

626. $\int \operatorname{cosec} x\, dx$

627. $\int \sec^2 (4x - 1)\, dx$

628. $\int \frac{\operatorname{tg} 3x}{\cos 3x}\, dx$

629. $\int \frac{dx}{\sin^2 x \cos^2 x}$

630. $\int \frac{dx}{\sin x \cos x}$

631. $\int \frac{\sin x}{\sqrt{\cos 2x}}\, dx$

632. $\int \frac{dz}{\sin z \cos^2 z}$

633. $\int \frac{dx}{\sqrt{x^2 + 11}}$

634. $\int \frac{\sqrt{x^2 + 1}}{x^2}\, dx$

635. $\int \sqrt{x^2 - 9}\, dx$

636. $\int \frac{x^2\, dx}{(x^2 + 4)^{3/2}}$

637. $\int \frac{dx}{\sqrt{x^2 - 2x + 17}}$

638. $\int \frac{dx}{x\sqrt{1 - x^2}}$

639. $\int \frac{dx}{(x^2 - 64)^3}$

640. $\int \sqrt{15 - 2x - x^2}\, dx$

641. $\int \sqrt{2 + x^2}\, dx$

642. $\int \frac{x^2\, dx}{\sqrt{9 + x^2}}$

643. $\int \sqrt{x^2 + x}\, dx$

644. $\int \frac{dx}{\sqrt{1 + 4x^2}}$

645. $\int \frac{dx}{\sqrt{6x - x^2}}$

646. $\int \frac{(x+3)\,dx}{\sqrt{x^2+8x+25}}$

647. $\int \sqrt{9x^2+6x+17}\,dx$

648. $\int \frac{x\,dx}{\sqrt{x^4+a^4}}$

649. $\int \sqrt{x^2+9}\,dx$

650. $\int \frac{dx}{(1+x^2)^{5/2}}$

651. $\int \frac{dx}{(1-x^2)\sqrt{1+x^2}}$

652. $\int \frac{x^7\,dx}{\sqrt{x^4-16}}$

653. $\int \frac{x^3\,dx}{(x^2-4)^3}$

654. $\int \sqrt{x^2-6x-7}\,dx$

655. $\int (x^2+x+1)^{3/2}\,dx$

656. $\int \frac{dx}{(x-1)\sqrt{x^2-3x+2}}$

657. $\int \frac{dx}{(x^2-2x+5)^{3/2}}$

658. $\int \frac{dx}{\sqrt{9+16x^2}}$

659. $\int \frac{dx}{\sqrt{-x^2-5x-4}}$

660. $\int (9x^2+6x+10)^{3/2}\,dx$

661. $\int \frac{4(x+1)\,dx}{\sqrt{4x^2-4x-3}}$

□ BAB IV
RUMUS-RUMUS REDUKSI

1. RUMUS REDUKSI UNTUK BINOMIAL DIFFERENSIAL

Bila binominal differensial tidak dapat diintegrasi dengan menggunakan rumus - rumus yang telah ada, maka diadakan rumus reduksi dengan menggunakan integrasi persiil.

Dibawah ini disajikan empat rumus reduksi yang pokok.

A). $$\int x^m (a + bx^n)^p \, dx = \frac{x^{m-n+1} (a + bx^n)^{p+1}}{(np + m + 1)\, b} - \frac{(m - n + 1)\, a}{(np + m + 1)b} \int x^{m-n} (a + bx^n)^p \, dx$$

B). $$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1} (a + bx^n)^p}{np + m + 1} + \frac{anp}{np + m + 1} \int x^m (a + bx^n)^{p-1} dx$$

C). $$\int x^m (a + bx^n)^p \, dx = \frac{x^{m+1} (a + bx^n)^{p+1}}{(m + 1)\, a} - \frac{(np + n + m + 1)\, b}{(m + 1)\, a} \int x^{m+n} (a + bx^n)^p \, dx$$

D). $$\int x^m (a + bx^n)^p \, dx = -\frac{x^{n+1} (a + bx^n)^{p+1}}{n\,(p + 1)\, a} + \frac{np + n + m + 1}{n\,(p + 1)\,a} \int x^m (a + bx^n)^{p+1} \, dx$$

Pada rumus - rumus diatas dapat dilihat, bahwa :

Rumus A) memperlihatkan bahwa m dikurangi dengan n , dan rumus A). tidak berlaku , jika np + m + 1 = 0

Rumus B) memperlihatkan bahwa p dikurangi dengan 1 , dan rumus B). tidak berlaku , jika np + m + 1 = 0

Rumus C) memperlihatkan bahwa m ditambah dengan n , dan rumus C). tidak berlaku , jika m + 1 = 0.

Rumus D) memperlihatkan bahwa p ditambah dengan 1 , dan rumus D). tidak berlaku , jika p + 1 = 0.

Contoh - contoh soal:

1. Tentukan : $\int \frac{x^3 \, dx}{\sqrt{1 - x^2}}$

Penyelesaian :

Disini digunakan rumus A). $\int \frac{x^3 \, dx}{\sqrt{1 - x^2}} = \int x^3 (1 - x^2)^{-\frac{1}{2}}$

Nampak disini m = 3

n = 2 dan p = $-\frac{1}{2}$

J a d i :

$$\int \frac{x^3 \, dx}{\sqrt{1 - x^2}} = \frac{x^{3 - 2 + 1} (1 - x^2)^{-\frac{1}{2} + 1}}{-1 (-1 + 3 + 1)}$$

$$- \frac{1 (3 - 2 + 1)}{-1 (-1 + 3 + 1)} \int x^{3 - 2} (1 - x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{1}{3} x^2 (1 - x^2)^{\frac{1}{2}} + \frac{2}{3} \int x (1 - x^2)^{-\frac{1}{2}} dx$$

$$= -\frac{1}{3} x^2 (1 - x^2)^{\frac{1}{2}} - \frac{2}{3} (1 - x^2)^{\frac{1}{2}} + C$$

$$= -\frac{1}{3} (x^2 + 2) (1 - x^2)^{\frac{1}{2}} + C$$

2). Tentukan : $\int \frac{x^4\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$

Penyelesaian :

Digunakan rumus A).

$\int \frac{x^4 dx}{(a^2 - x^2)^{\frac{1}{2}}} = x^4 (a^2 - x^2)^{-\frac{1}{2}} dx$, disini m = 4 , n = 2 dan p = $-\frac{1}{2}$

J a d i :

$$\int \frac{x^4\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = \frac{x^{4-2+1}(a^2 - x^2)^{-\frac{1}{2}+1}}{-1\,(-1+4+1)} - \frac{(4-2+1)\,a^2}{-1\,(-1+4+1)} \int x^{4-2}(a^2 - x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{x^3 (a^2 - x^2)^{\frac{1}{2}}}{4} + \frac{3}{4} a^2 \int x^2 (a^2 - x^2)^{-\frac{1}{2}}\,dx$$

$$= -\frac{x^3 (a^2 - x^2)^{\frac{1}{2}}}{4} + \frac{3}{4} a^2 \left[\frac{x^{2-2+1}(a^2 - x^2)^{\frac{1}{2}}}{-1\,(-1+2+1)} \right] - \frac{3}{4} a^2 \left[\frac{2-2+1}{-1\,(-1+2+1)} \int x^{2-2}(a^2 - x^2)^{-\frac{1}{2}}\,dx \right]$$

$$= -\frac{x^3 (a^2 - x^2)^{\frac{1}{2}}}{4} - \frac{3}{8} a^2 (a^2 - x^2)^{\frac{1}{2}} + \frac{3}{8} a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= -(\tfrac{1}{4} x^3 + \tfrac{3}{8} a^2) \sqrt{a^2 - x^2} + \frac{3}{8} a^2 \arcsin \frac{x}{a} + C.$$

3). Tentukan : $\int (a^2 + x^2)^{\frac{1}{2}}\,dx$

Penyelesaian :

Disini digunakan rumus B).

Bentuk $\int (a^2 + x^2)^{\frac{1}{2}}\,dx$ mempunyai m = 0, n = 2 dan p = $\frac{1}{2}$,

M a k a :

$$\int (a^2 + x^2)^{\frac{1}{2}}\, dx = \frac{x^{o+1}(a^2 + x^2)^{\frac{1}{2}}}{1 + 0 + 1} + \frac{a^2}{1 + 0 + 1}\int x^o (a^2 + x^2)^{\frac{1}{2}-1} dx$$

$$= \frac{x\,(a^2 + x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2} \int \frac{dx}{\sqrt{a^2 + x^2}}$$

$$= \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \ln\,(x + \sqrt{a^2 + x^2}) + C.$$

4). Tentukan : $\int \frac{dx}{x^3\sqrt{x^2 - 1}}$

Penyelesaian :

Digunakan rumus C).

Dari bentuk $\int \frac{dx}{x^3\sqrt{x^2 - 1}} = \int x^{-3}\,(x^2 - 1)^{-\frac{1}{2}}\, dx$ didapat

$m = -3$, $n = 2$ dan $p = -\frac{1}{2}$

J a d i :

$$\int \frac{dx}{x^3\sqrt{x^2 - 1}} = \frac{x^{-3+1}\,(x^2 - 1)^{-\frac{1}{2}+1}}{-1\,(-3+1)}$$

$$- \frac{(-1 + 2 - 3 + 1)}{-1\,(-3+1)} \int x^{-3+2}(x^2 - 1)^{-\frac{1}{2}} dx$$

$$= \frac{\sqrt{x^2 - 1}}{2x^2} + \tfrac{1}{2}\int \frac{dx}{x\sqrt{x^2 - 1}}$$

$$= \frac{\sqrt{x^2 - 1}}{2x^2} + \tfrac{1}{2}\ \text{arc sec}\ x + C.$$

Soal - soal

662. $\int \frac{x^2\, dx}{\sqrt{a^2 - x^2}}$

663. $\int (x^2 + a^2)^{3/2}\, dx$

664. $\int \frac{\sqrt{a^2 + x^2}}{x}\, dx$

665. $\int \frac{a^2 + x^2}{t^3\sqrt{1 - 4t^2}}$

666. $\int \frac{\sqrt{a^2 - x^2}}{x^4}\,dx$

667. $\int y^2 \sqrt{4 - 9y^2}\,dy$

668. $\int \frac{dx}{(1 + 4x^2)^2}$

669. $\int \frac{x^2\,dx}{(a^2 - x^2)^2}$

670. $\int \frac{x^7\,dx}{(3 + 5x^4)^{2/3}}$

671. $\int \frac{x^2\,dx}{\sqrt{2ax - x^2}}$

672. $\int \frac{ds}{(a^2 + s^2)^3}$

673. $\int \frac{dx}{(a^2 - x^2)^{5/2}}$

674. $\int \frac{dx}{(a^2 + x^2)^2}$

675. $\int \frac{dx}{(x^2 - 16)^{3/2}}$

676. $\int (25 - x^2)^{3/2}\,dx$

677. $\int \frac{dx}{(25 - 16x^2)^{3/2}}$

678. $\int \frac{4\,dx}{\sqrt{16x^2 - 40x + 50}}$

679. $\int \frac{x^3\,dx}{\sqrt{a^2 + x^2}}$

680. $\int (9x^2 + 4)^{3/2}\,dx$

681. $\int \frac{(1 - x^3)^{5/2}}{x}\,dx$

682. $\int \frac{z^8\,dz}{\sqrt{5 - z^3}}$

683. $\int \frac{\sqrt{a^2 + x^2}}{a^2}\,dx$

684. $\int x^2 \sqrt{a^2 - x^2}\,dx$

685. $\int \frac{1}{x^2 (1 + x^2)^{3/2}}\,dx$

686. $\int \frac{x^8}{\sqrt{4 - x^6}}\,dx$

687. $\int \frac{y^3\,dy}{\sqrt{4y - y^2}}$

688. $\int \frac{t^3\,dt}{\sqrt{1 + 4t^2}}$

689. $\int \frac{x^5\,dx}{\sqrt{1 - x^3}}$

690. $\int \frac{x^3\,dx}{(1 + 9x^2)^{3/2}}$

691. $\int \frac{dx}{x^3 \sqrt{a^2 - x^2}}$

692. $\int \frac{dx}{(x^2 - 16)^{5/2}}$

693. $\int (4 - x^2)^{5/2}\,dx$

694. $\int \frac{dx}{(16 + 25\,x^2)^{5/2}}$ 695. $\int \frac{3\,dx}{\sqrt{9\,x^2 + 6\,x + 8}}$

2. RUMUS REDUKSI UNTUK TRIGONOMETRIK DIFFERENSIAL

Rumus reduksi untuk trigonometri :

E). $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x\,dx$

F). $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x\,dx$

G). $\int \sin^m x \cos^n x\,dx = -\frac{\sin^{m+1} x \cos^{n+1} x}{n+1} + \frac{m+n+2}{n+1} \int \sin^m x \cos^{n+2} x\,dx$

H). $\int \sin^m x \cos^n x\,dx = \frac{\sin^{m+1} x \cos^{n+1}}{m+1} + \frac{m+n+2}{m+1} \int \sin^{m+2} x \cos^n x\,dx$

Pada rumus - rumus diatas dapat dilihat , bahwa :

Rumus E). memperlihatkan bahwa n dikurangi dengan 2 , dan rumus E). tidak berlaku , jika $m + n = 0$.

Rumus F). memperlihatkan bahwa m dikurangi dengan 2 , dan rumus F). tidak berlaku , jika $m + n = 0$.

Rumus G). memperlihatkan bahwa n ditambah dengan 2 , dan rumus G). tidak berlaku , jika $n + 1 = 0$.

Rumus H). memperlihatkan bahwa m ditambah dengan 2, dan rumus H). tidak berlaku , jika $m + 1 = 0$.

Contoh - contoh soal :

5). Tentukan : $\int \sin^2 x \cos^4 x \, dx$

Penyelesaian :

Disini digunakan rumus F).

Untuk bentuk $\sin^2 x \cos^4 x \, dx$ terdapat m = 2 , n = 4 dan m + n = 2 + 4 = 6 ≠ 0 , maka integral dapat diselesaikan dengan rumus F).

$$\int \sin^2 x \cos^4 x \, dx = -\frac{\sin^{2-1} x \cos^{4+1} x}{2+4} + \frac{2-1}{2+4}\int \sin^{2-2} x \cos^4 x \, dx$$

$$= -\frac{\sin x \cos^5 x}{6} + \frac{1}{6}\int \cos^4 x \, dx$$

Sedang untuk $\int \cos^4 x \, dx$ digunakan rumus E). , yaitu m = 0 n = 4 , maka :

$$\int \cos^4 x \, dx = \frac{\sin^{0+1} x \cos^{4-1} x}{0+4} + \frac{4-1}{0+4}\int \sin^0 x \cos^{4-2} x \, dx$$

$$= \frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x \, dx$$

$$= \frac{\sin x \cos^2 x}{4} + \frac{3}{8} \sin x \cos x + \frac{3}{8}\int dx$$

Jadi hasil Integral seluruhnya :

$$\int \sin^2 x \cos^4 x \, dx = -\frac{\sin x \cos^5 x}{6} + \frac{\sin x \cos^3 x}{24} + \frac{1}{16}(\sin x \cos x + x) + C.$$

6). Tentukan : $\int \frac{tg^2 2x}{\cos 2x} \, dx$

Penyelesaian :

Bentuk $\frac{tg^2\ 2x}{\cos\ 2x} = \frac{\sin^2\ 2x}{\cos^3\ 2x} = \sin^2\ 2x\ \cos^{-3}\ 2x$, dan dx harus disesuaikan, maka andaikan : $2x = U$ dan $dx = \frac{1}{2}\ dU$.

Bentuk integral sekarang adalah :

$\frac{1}{2}\int \sin^2 U \cos^{-3} U\ dU$, untuk bentuk ini digunakan rumus G), terdapat :

$$\frac{1}{2}\int \sin^2 U \cos^{-3} U\ dU = -\frac{\sin^3 U \cos^{-2} U}{-4} + \frac{1}{-4}\int \sin^2 U \cos^{-1} U\ dU$$

Untuk menyelesaikan integralnya , digunakan rumus F), terdapat :

$$= +\frac{\sin^3 U \cos^{-2} U}{4} + \tfrac{1}{4}\sin U - \tfrac{1}{4}\int \cos^{-1} U\ dU$$

$$= \frac{\sin^3 U \sec^2 U}{4} + \tfrac{1}{4}\sin U - \tfrac{1}{4}\ln(\sec U + tg\ U) + C$$

Setelah u diganti, terdapat :

$$\int \frac{tg^2\ 2x}{\cos\ 2x}\ dx = \tfrac{1}{4}\sin^3 2x \sec^2 2x + \tfrac{1}{4}\sin 2x - \tfrac{1}{4}\ln (\sec 2x + tg\ 2x) + C.$$

7). Tentukan : $\int \sin^3 x \cos^{2/3} x\ dx$

Penyelesaian :

Untuk integral diatas ini, digunakan rumus F).

$$\int \sin^3 x \cos^{2/3} x\ dx = -\frac{\sin^2 x \cos^{5/3} x}{11/3} + \frac{2}{11/3}\int \sin x \cos^{2/3} x\ dx$$

$$= -\frac{3}{11}\sin^2 x \cos^{5/3} x - \frac{18}{55}\cos^{5/3} x + \frac{1-1}{1+2/3}\int \sin^{1-2} x \cos^{2/3} x\ dx.$$

$$= -\frac{3}{11}\cos^{5/3} x\ (\sin^2 x + \frac{6}{5}) + C.$$

8). Tentukan : $\int \sec^3 x\ dx$, dengan menggunakan rumus reduksi.

Penyelesaian :

Bentuk $\int \sec^3 x\, dx = \int \cos^{-3} x\, dx$, disini m = 0 , n = - 3 dan digunakan rumus G) , yaitu :

$$\int \sec^3 x\, dx = -\frac{\sin x \cos^{-2} x}{-2} + \frac{-3+2}{-3+1}\int (\cos x)^{-1}\, dx$$

$$= \tfrac{1}{2} \operatorname{tg} x \sec x + \tfrac{1}{2} \ln \left| \sec x + \operatorname{tg} x \right| + C.$$

Soal - soal :

696. $\int \sin^4 x \cos^2 x\, dx$

697. $\int \operatorname{cosec}^5 x\, dx$

698. $\int \operatorname{ctg}^4 \theta\, d\theta$

699. $\int \sin^2 \phi \cos^2 \phi\, d\phi$

700. $\int \operatorname{ctg}^2 x \cos x\, dx$

701. $\int \operatorname{cosec}^3 \frac{x}{2}\, dx$

702. $\int \sec^5 x\, dx$

703. $\int \frac{d\phi}{\sin^4 \phi \cos^2 \phi}$

704. $\int \frac{\sin^7 x}{\cos^5 x}\, dx$

705. $\int \cos^6 x\, dx$

706. $\int \sec^4 x\, dx$

707. $\int \sin^3 x\, dx$

708. $\int \sin^6 2x\, dx$

709. $\int \sin^4 x \cos^5 x\, dx$

710. $\int \sin^{\frac{1}{2}} \phi \cos^3 \phi\, d\phi$

711. $\int \operatorname{cosec}^3 x\, dx$

712. $\int \sin^3 x \cos^{4/5} x\, dx$

713. $\int \frac{dx}{\cos x \sin^5 x}$

714. $\int \frac{\operatorname{ctg}^2 2x\, dx}{\sin 2x}$

715. $\int \sin^{-4} x\, dx$

□ BAB V
PENYELESAIAN SOAL-SOAL UJIAN

PENYELESAIAN SOAL-SOAL UJIAN (CAMPURAN)

1). Tentukan : $\int \frac{dx}{\sqrt{x^2 + 4x + a}}$, untuk a = 5 untuk a =3

(ujian tahun 1915).

Penyelesaian :

a). Untuk a = 5 , bentuk $x^2 + 4x + 5 = (x + 2)^2 + 1$, maka :

$$\int \frac{dx}{\sqrt{x^2 + 4x + 5}} = \int \frac{dx}{\sqrt{(x + 2)^2 + 1}}$$

$$= \ln \left| x + 2 + \sqrt{x^2 + 4x + 5} \right| + C.$$

b). Untuk a = 3 , bentuk $x^2 + 4x + 3 = (x + 2)^2 - 1$, maka :

$$\int \frac{dx}{\sqrt{x^2 + 4x + 3}} = \int \frac{dx}{\sqrt{(x + 2)^2 - 1}}$$

$$= \ln \left| x + 2 + \sqrt{x^2 + 4x + 3} \right| + C.$$

2). Tentukan : $\int \frac{72\, dx}{(x - 1)(x + 5)^2}$ (ujian tahun 1915)

Penyelesaian :

Pandang dulu bentuk :

$$\frac{72}{(x-1)(x+5)^2} = \frac{A}{x-1} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

$$72 = A(x+5)^2 + B(x-1)(x+5) + C(x-1)$$

Untuk $x = 1$, $72 = 36A$ dan $A = 2$

$x = -5$, $72 = -6C$ dan $C = -12$

Dengan identitas , terdapat $A + B = 0$, maka $B = -2$

J a d i :

$$\int \frac{dx}{(x-1)(x+5)^2} = 2\int \frac{dx}{x-1} - 2\int \frac{dx}{x+5} - 12\int \frac{dx}{(x+5)^2}$$

$$= 2 \ln (x-1) - 2 \ln (x+5) + \frac{12}{x+5} + C.$$

$$= 2 \ln \left| \frac{x-1}{x+1} \right| + \frac{12}{x+5} + C.$$

3). Tentukan : $\int \frac{5\,dx}{3 \operatorname{tg} x + 2 \cos x}$ (Ujian tahun 1918)

Penyelesaian :

$$\int \frac{5\,dx}{3 \operatorname{tg} x + 2 \cos x} = \int \frac{5\,dx}{3 \frac{\sin x}{\cos x} + 2 \cos x}$$

$$= \int \frac{5 \cos x\,dx}{3 \sin x + 2 \cos^2 x} = \int \frac{5 \cos x\,dx}{3 \sin x + 2 - 2 \sin^2 x}$$

$$= \frac{5}{2} \int \frac{d(\sin x)}{\frac{25}{16} - (\sin x - \frac{3}{4})^2}$$

$$= \ln \left| \frac{\frac{5}{4} + \sin x - \frac{3}{4}}{\frac{5}{4} - \sin x + \frac{3}{4}} \right| + C.$$

$$= \ln \left| \frac{1 + 2 \sin x}{4 - 2 \sin x} \right| + C$$

4). Tentukan : $\int \sin 2x \cdot \ln \sin x \, dx$ (ujian tahun 1919)

Penyelesaian :

Untuk menyelesaikan integral , digunakan integral persiil :

$$\int \sin 2x \cdot \ln \sin x \, dx = \int 2 \sin x \cos x \cdot \ln \sin x \, dx$$

$$= 2 \int \sin x \ln \sin x \, d(\sin x) = \int \ln \sin x \, d(\sin^2 x)$$

$$= \sin^2 x \ln \sin x - \int \sin^2 x \, d(\ln \sin x)$$

$$= \sin^2 x \ln \sin x - \int \sin^2 x \cdot \frac{\cos x}{\sin x} \, dx$$

$$= \sin^2 x \ln \sin x - \tfrac{1}{2} \sin^2 x + C$$

$$= \tfrac{1}{2} \sin^2 x (2 \ln \sin x - 1) + C.$$

5). Tentukan : $\int \sin x \cos x (\cos 3x + e^{\sin x}) \, dx$ (ujian tahun 1920)

Penyelesaian :

Bentuk diatas ditulis menjadi jumlah dua integral.

$$\int \sin x \cos x (\cos 3x + e^{\sin x}) \, dx = \int \sin x \cos x \cos 3x \, dx + \int \sin x \cos x \, e^{\sin x} \, dx.$$

$$= \tfrac{1}{2} \int \sin 2x \cos 3x \, dx + \int \sin x \, e^{\sin x} \, d(\sin x)$$

$$= \tfrac{1}{4} \int \left[\sin (2x + 3x) + \sin (2x - 3x) \right] dx + \int \sin x \, d(e^{\sin x})$$

$$= \tfrac{1}{4} \int \left[\sin 5x - \sin x \right] dx + \sin x \, e^{\sin x} - \int e^{\sin x} \, d\sin x$$

$$= \frac{1}{20}\left[5 \cos x - \cos 5x \right] + \sin x\, e^{\sin x} - e^{\sin x} + C.$$

6) Tentukan : $\int \frac{a + \sqrt{a - x}}{x}\, dx$ (ujian tahun 1920)

Penyelesaian :

Diambil substitusi : $a - x = U^2$, $x = a - U^2$ dan $dx = -2\, U\, dU$,

M a k a :

$$\int \frac{a + \sqrt{a - x}}{x}\, dx = \int \frac{a + U}{a - U^2}(-2\, U\, dU) = 2\int \frac{a\, dU}{U^2 - a} + 2\int \frac{U^2\, dU}{U^2 - a}$$

$$= \sqrt{a}\, \ln \left| \frac{U - a}{U + a} \right| + 2\int dU + 2a \int \frac{U\, dU}{U^2 - a}$$

$$= \sqrt{a}\, \ln \left| \frac{U - a}{U + a} \right| + 2U + a \ln (U^2 - a) + C$$

Setelah u diganti, terdapat :

$$\int \frac{a + \sqrt{a - x}}{x}\, dx = 2\sqrt{a - x} + a \ln x + \sqrt{a} \ln \left| \frac{\sqrt{a - x} - \sqrt{a}}{\sqrt{a - x} + \sqrt{a}} \right| + C.$$

7). Tentukan : $\int \left(\frac{x^3 + 1}{x} \ln x + \frac{1}{3 + \sin x}\right) dx$ (ujian tahun 1921)

Penyelesaian :

Bentuk integral diatas dapat ditulis sebagai , jumlahan dua buah integral , yaitu :

$$\int \left(\frac{x^3 + 1}{x} \ln x + \frac{1}{3 + \sin x}\right) dx = \int \frac{x^3 + 1}{x} \ln x\, dx + \int \frac{dx}{3 + \sin x}$$

Dicari dulu masing-masing integral :

a). $\int \frac{x^3 + 1}{x} \ln x\, dx = \int x^2 \ln x\, dx + \int \frac{1}{x} \ln x\, dx$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3}\int x^3 \cdot \frac{1}{x}\, dx + \int \ln x \, d(\ln x)$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + \tfrac{1}{2} \ln^2 x + C.$$

b). $\int \frac{dx}{3 + \sin x}$, untuk ini diambil substitusi $U = \operatorname{tg} \frac{x}{2}$ dan

$x = 2 \operatorname{arc\,tg} U$, $dx = \frac{2\, dU}{1 + U^2}$; $\sin x = \frac{2U}{1 + U^2}$, maka

$$\int \frac{dx}{3 + \sin x} = \int \frac{\frac{2\, dU}{1 + U^2}}{3 + \frac{2U}{1 + U^2}} = \int \frac{2\, dU}{3U^2 + 2U + 3}$$

$$= \frac{2}{3}\int \frac{dU}{U^2 + \frac{2}{3} U + 1} = \frac{2}{3} \int \frac{dU}{(U + \frac{1}{3})^2 + \frac{8}{9}}$$

$$= \tfrac{1}{2} \sqrt{2} \operatorname{arc\,tg} \frac{3U + 1}{2\sqrt{2}} + C$$

$$= \tfrac{1}{2} \sqrt{2} \operatorname{arc\,tg} \left(\frac{3 \operatorname{tg} \frac{x}{2} + 1}{2\sqrt{2}}\right) + C.$$

Jadi , integral seluruhnya , adalah :

$$\int \left(\frac{x^3 + 1}{x} \ln x + \frac{1}{3 + \sin x}\right) dx = \frac{1}{9} x^3 (3 \ln x - 1) + \tfrac{1}{2} \ln^2 x$$

$$+ \tfrac{1}{2} \sqrt{2} \operatorname{arc\,tg} \left(\frac{3 \operatorname{tg} \frac{x}{2} + 1}{2\sqrt{2}}\right) + C.$$

8). Tentukan : $\int \frac{x \arcsin x + x^2 + 1}{\sqrt{(1 - x^2)^3}}\, dx$ (ujian tahun 1921)

Penyelesaian :

$$\int \frac{x \arcsin x + x^2 + 1}{\sqrt{(1 - x^2)^3}}\, dx = \int \frac{x \arcsin x}{\sqrt{(1 - x^2)^3}}\, dx + \int \frac{x^2 + 1}{\sqrt{(1 - x^2)^3}}\, dx$$

Dicari dulu satu persatu :

a). $\int \frac{x \arcsin x}{\sqrt{(1 - x^2)^3}}\, dx = \int \arcsin x \cdot d\sqrt{(1 - x^2)^{-1}}$

$= \arcsin x \cdot \sqrt{(1 - x^2)^{-1}} -$

$- \int (1 - x^2)^{-\frac{1}{2}} \cdot \frac{1}{(1 - x^2)^{\frac{1}{2}}}\, dx$

$= \frac{\arcsin x}{\sqrt{1 - x^2}} + \int \frac{dx}{x^2 - 1}$

$= \frac{\arcsin x}{\sqrt{1 - x^2}} + \frac{1}{2} \ln \frac{x - 1}{x + 1} + C_1$

b). $\int \frac{x^2 + 1}{\sqrt{(1 - x^2)^3}}\, dx$, diambil substitusi $x = \sin U$, $dx = \cos U\, dU$

M a k a integral menjadi :

$= \int \frac{\sin^2 U + 1}{\sqrt{(1 - \sin^2 U)^3}} \cos U\, dU = \int \frac{\sin^2 U + 1}{\cos^2 U}\, dU$

$= \int \frac{2 - \cos^2 U}{\cos^2 U}\, dU = 2 \int \sec^2 U\, dU - \int dU$

$= 2 \operatorname{tg} U - U + C.$

Jadi integral setelah U diganti dengan fungsi x , terdapat :

$$\int \frac{x^2 - 1}{\sqrt{(1 - x^2)^3}}\, dx = \frac{2x}{\sqrt{1 - x^2}} - \arcsin x + C.$$

Hasil seluruhnya , adalah :

$$\int \frac{x \arcsin x + x^2 + 1}{\sqrt{(1 - x^2)^3}}\, dx = \frac{\arcsin x + 2x}{\sqrt{(1 - x^2)}} - \arcsin x$$

$$+ \tfrac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.$$

9). Tentukan : $\int \frac{dx}{(1 + x)^2 (1 - x^3)}$ (ujian tahun 1922).

Penyelesaian :

$$\int \frac{dx}{(1 + x)^2 (1 - x^3)} = \int \frac{dx}{(1 - x) (1 + x)^2 (1 + x + x^2)}$$

Pandang dulu bentuk :

$$\frac{1}{(1 - x) (1 + x)^2 (1 + x + x^2)} = \frac{A}{(1 - x)} + \frac{B}{1 + x} + \frac{C}{(1 + x)^2} + \frac{Dx + E}{1 + x + x^2}$$

$$1 = A (1 + x)^2 (1 + x + x^2) + B (1 - x) (1 + x) (1 + x + x^2)$$

$$+ C (1 - x) (1 + x + x^2) + (D x + E) (1 - x) (1 + x)^2$$

Untuk : X = 1 , maka 1 = 12 A dan $A = \frac{1}{12}$

x = - 1 , maka 1 = 2 C dan C = ½

Hasil ini dimasukkan ke persamaan , kemudian disederhanakan :

$$1 - \frac{1}{12} (1 + x)^2 (1 + x + x^2) - \tfrac{1}{2} (1 - x) (1 + x + x^2) =$$

$$= (1 - x) (1 + x) \left[B (1 + x + x^2) + (D x + E) (1 + x) \right]$$

$$- \frac{1}{12} (x^2 - 3 x + 5) (1 + x) (1 - x) = (1 - x) (1 + x)$$

$$\left[B (1 + x + x^2) + (D x + E) (1 + x) \right] .$$

$$\frac{1}{12} (x^2 - 3 x + 5) = B (1 + x + x^2) + (Dx + E) (1 + x)$$

Untuk : x = - 1 , maka $\frac{9}{12} = B$ dan $B = \frac{3}{4}$

Dengan identiteit :

Koeffisien x^2 : $B + D = \frac{1}{12}$ dan $D = -\frac{8}{12} = -\frac{2}{3}$

Koeffisien x^0 : $B + E = \frac{5}{12}$ dan $E = -\frac{4}{12} = -\frac{1}{3}$

Maka , integral sekarang :

$$\int \frac{dx}{(1+x)^2 (1-x^3)} = \frac{1}{12}\int \frac{dx}{1-x} + \frac{3}{4}\int \frac{dx}{1+x} + \tfrac{1}{2}\int \frac{dx}{(1+x)^2}$$

$$- \frac{1}{3}\int \frac{2x+1}{1+x+x^2}\,dx$$

$$= \frac{1}{12}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2(1+x)} - \frac{1}{3}\int \frac{d(x^2+x)}{1+x+x^2}$$

$$= \frac{1}{12}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2(x+1)} - \frac{1}{3}\ln(1+x+x^2) + C$$

$$= \frac{1}{12}\ln\left|\frac{x+1}{(x-1)(x^2+x+1)^4}\right| - \frac{1}{2(x+1)} + C$$

10). Selidikilah akar - akar dari persamaan :

$$x^4 + 4x^3 - 16x - 16 = 0$$

dan hitunglah integral :

$$\int \frac{(x^2-2)(4x^3+x^2+2)}{x^8+4x^6-16x^2-16}\,dx$$

Penyelesaian :

$$x^4 + 4x^3 - 16x - 16 = 0$$

$$x^4 - 16 + 4x(x^2 - 4) = 0$$

$$(x^2-4)(x^2+4) + 4x(x^2-4) = 0$$

$$(x^2-4)(x^2+4x+4) = 0$$

$$(x - 2)\,(x + 2)^3 = 0$$

Jadi akar - akar dari persamaan ialah :

$x_1 = 2\,,\quad x_2 = x_3 = x_4 = -2$

Untuk hitungan integral :

$$\int \frac{(x^2 - 2)\,(4x^3 + x^2 + 2)}{x^8 + 4x^6 - 16x^2 - 16}\,dx = \int \frac{(x^2 - 2)\,(4x^3 - x^2 + 2)}{(x^2 - 2)\,(x^2 + 2)^3}\,dx$$

$$= \int \frac{4x^3 + x^2 + 2}{(x^3 + 2)^3}\,dx$$

Pandang dulu bentuk :

$$\frac{4x^3 + x^2 + 2}{(x^2 + 2)^3} - \frac{Ax + B}{x^2 + 2} + \frac{Cx + D}{(x^2 + 2)^2} + \frac{Ex + F}{(x^2 + 2)^3}$$

$$4x^3 + x^2 + 2 = (Ax + B)\,(x^2 + 2)^2 + (Cx + D)\,(x^2 + 2) +$$

$$+ (Ex + F).$$

Digunakan identitas :

koeffisien x^5 : $0 = A$ dan $A = 0$

koeffisien x^4 : $0 = B$ dan $B = 0$

koeffisien x^3 : $4 = 4A + C$ dan $C = 4$

koeffisien x^2 : $1 = 4B + D$ dan $D = 1$

koeffisien x^1 : $0 = 4A + 2C + E$ dan $E = -8$

koeffisien x^o : $2 = 4B + 2D + F$ dan $F = 0$

M a k a :

$$\int \frac{4x^3 + x^2 + 2}{(x^2 + 2)^3}\,dx = \int \frac{4x + 1}{(x^2 + 2)^2}\,dx - 8\int \frac{x\,dx}{(x^2 + 2)^3}$$

$$= 2\int \frac{d\,(x^2+2)}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2} - 4\int \frac{d\,(x^2+2)}{(x^2+2)^3}$$

$$= -\frac{2}{x^2+2} + \frac{2}{(x^2+2)^2} + \int \frac{dx}{(x^2+2)^2}$$

Dicari dulu bentuk : $\int \frac{dx}{(x^2+2)^2}$. misalkan $x = \sqrt{2}\ \text{tg}\ U$

$dx = \sqrt{2}\ \sec^2 U\ dU$

M a k a : $\int \frac{dx}{(x^2+2)^2} = \int \frac{\sqrt{2}\sec^2 U}{2^2 \sec^4 U}\, dU$

$$= \tfrac{1}{4}\sqrt{2}\int \cos^2 U\, dU = \frac{1}{8}\sqrt{2}\int (1 + \cos 2U)\, dU$$

$$= \frac{1}{8}\sqrt{2}\,(U + \tfrac{1}{2}\sin 2U) + C$$

$$= \frac{1}{8}\sqrt{2}\left(\text{arc tg}\frac{x}{\sqrt{2}} + \frac{x\sqrt{2}}{x^2+2}\right) + C.$$

Jadi hasil integral seluruhnya , adalah :

$$\int \frac{(x^2-2)(4x^3+x^2+2)}{x^8+4x^6-16x^2-16}\,dx = -\frac{2}{x^2+2} + \frac{2}{(x^2+2)^2}$$

$$+ \frac{x}{4(x^2+2)} + \frac{1}{8}\sqrt{2}\,\text{arc tg}\,\frac{x}{\sqrt{2}} + C.$$

11). Tentukan : $\int \left[\frac{1}{x^2(x-1)} - \frac{1}{x^2\sqrt{x-1}}\right] dx$

(ujian tahun 1923).

Penyelesaian :

Diambil substitusi : $x - 1 = \text{tg}\ U$, $x = 1 + \text{tg}^2 U = \sec^2 U$

$dx = 2\sec^2 U\ \text{tg}\ U\ dU$, maka :

$$\int \left[\frac{1}{x^2(x-1)} - \frac{1}{x^2\sqrt{x-1}}\right] dx =$$

$$= \int \frac{2 \sec^2 U \operatorname{tg} U}{\sec^4 U \cdot \operatorname{tg}^2 U} dU - \int \frac{2 \sec^2 U \operatorname{tg} U}{\sec^4 U \cdot \operatorname{tg} U} dU$$

$$= 2 \int \frac{\cos^3 U}{\sin U} dU - 2 \int \cos^2 U \, dU$$

$$= 2 \int (\operatorname{cosec} U - \sin U) \, d(\sin U) - \int (1 + \cos 2U) \, dU$$

$$= 2 \ln \sin U - \sin^2 U - U - \tfrac{1}{2} \sin 2U + C.$$

$$= \ln \frac{x-1}{x} - \frac{x-1}{x} - \operatorname{arc\,tg} \sqrt{x-1} - \frac{\sqrt{x-1}}{x} + C.$$

12). Tentukan : $\int \frac{x^3 - 16x^2 - 55x + 46}{x^4 - 37x^2 - 74x - 70} dx$ (ujian tahun 1923)

Penyelesaian :

Bentuk $x^4 - 37x^2 - 74x - 70$ ditulis menjadi perkalian faktor faktor. Untuk mencari faktor-faktor ini, kita lakukan demikian. Pandang koeffisien-koeffisien mulai x^4 ber turut-turut sampai x^0 , dan ditulis berurutan :

1	0	- 37	-74	-70	|- 5
	- 5	25	60	70	
1	- 5	- 12	-14	|0	|7
	+ 7	+ 14	+ 14		
1	2	2	|0		

Jadi faktor-faktornya adalah : $(x + 5)(x - 7)(x^2 + 2x + 2)$

Dan bentuk $\frac{x^3 - 16x^2 - 55x + 46}{x^4 - 37x^2 - 74x - 70} = \frac{x^3 - 16x^2 - 55x + 46}{(x+5)(x-7)(x^2+2x+2)}$

$$= \frac{A}{x+5} + \frac{B}{x-7} + \frac{Cx + D}{x^2 + x + 2}$$

$$x^3 - 16x^2 - 55x + 46 = A(x-7)(x^2+2x+2) + B(x+5)(x^2+2x+2) + (x+5)(x-7)(Cx+D).$$

Untuk $x = -5$, terdapat $-204 = A(-12)(17)$ dan $A = 1$

$x = 7$, terdapat $-780 = B(12)(65)$ dan $B = -1$

Dengan identiteit :

koeffisien x^3 : $A + B + C = 1$, maka $C = 1$

koeffisien x^0 : $-14A + 10B - 35D = 46$, maka $D = -2$

Jadi integral berbentuk :

$$\int \frac{x^3 - 16x^2 - 55x + 46}{x^4 - 37x^2 - 74x - 70}\,dx = \int \frac{dx}{x+5} - \int \frac{dx}{x-7} + \int \frac{x-2}{x^2+2x+2}\,dx$$

$$= \ln(x+5) - \ln(x-7) + \int \frac{(x+1)\,dx}{x^2+2x+2} - \int \frac{3\,dx}{(x+1)^2+1}$$

$$= \ln\left|\frac{x+5}{x-7}\right| + \tfrac{1}{2}\ln(x^2+x+2) - 3\,\text{arc tg}(x+1) + C.$$

13). Tentukan : $\int \frac{\left(\sqrt{2\sin\theta - \cos^2\theta} + \sqrt{2\sin\theta + \cos^2\theta}\right)\cos\theta}{\sqrt{4\sin^2\theta - \cos^4\theta}}\,d\theta$

(ujian tahun 1923).

Penyelesaian :

$$\int \frac{\sqrt{2\sin\theta - \cos^2\theta} + \sqrt{2\sin\theta + \cos^2\theta\cos\theta}}{\sqrt{4\sin^2\theta - \cos^4\theta}}\,d\theta$$

$$= \int \frac{\sqrt{2\sin\theta - \cos^2\theta} + \sqrt{2\sin\theta + \cos^2\theta\cos\theta}}{\sqrt{(2\sin\theta + \cos^2\theta)(2\sin\theta - \cos^2\theta)}}\,d\theta$$

$$= \int \frac{\cos\theta\,d\theta}{\sqrt{2\sin\theta + \cos^2\theta}} + \int \frac{\cos\theta\,d\theta}{\sqrt{2\sin\theta - \cos^2\theta}}$$

$$= \int \frac{d(\sin\theta)}{\sqrt{1 + 2\sin\theta - \sin^2\theta}} + \int \frac{d(\sin\theta)}{\sqrt{\sin^2\theta + 2\sin\theta - 1}}$$

$$= \int \frac{d(\sin\theta)}{\sqrt{2 - (\sin\theta - 1)^2}} + \int \frac{d(\sin\theta)}{\sqrt{(\sin\theta + 1)^2 - 2}}$$

$$= \arcsin\left(\frac{\sin\theta - 1}{\sqrt{2}}\right) + \ln\left|\sin\theta + 1 + \sqrt{\sin^2\theta + 2\sin\theta - 1}\right|$$

$+ C.$

14). Tentukan : $\int \frac{\sin x}{\cos^2 x \sqrt{4 - 5\sin^2 x}}\, dx$ (ujian th. 1924).

Penyelesaian :

$$\int \frac{\sin x}{\cos^2 x \sqrt{4 - 5\sin^2 x}}\, dx = \int \frac{\sin x\, dx}{\cos^2 x \sqrt{4 - 5(1 - \cos^2 x)}}$$

$$= \int \frac{\sin x}{\cos^2 x \sqrt{5\cos^2 x - 1}}\, dx$$

Diambil substitusi : $\cos x = \frac{1}{\sqrt{5}} \sec U$ dan

$$-\sin x\, dx = \frac{1}{\sqrt{5}} \sec U \cdot \operatorname{tg} U \cdot dU$$

Maka , bentuk integral menjadi :

$$= \int \frac{-\frac{1}{\sqrt{5}} \sec U \operatorname{tg} U\, dU}{\frac{1}{5}\sec^2 U \sqrt{\sec^2 U - 1}} = \frac{5}{\sqrt{5}} \int \sec U\, dU$$

$$= -\sqrt{5}\sin U + C.$$

Jadi :

$$\int \frac{\sin x\, dx}{\cos^2 x \sqrt{4 - 5\sin^2 x}} = -\frac{\sqrt{5\cos^2 x - 1}}{\cos x} + C \quad \text{atau}$$

$$= -\frac{\sqrt{4 - 5\sin^2 x}}{\cos x} + C.$$

15). Tentukan : $\int \frac{x\,(1+x)}{(1+2x+2x^2)^2}\,dx$

Penyelesaian :

Pandang dulu bentuk :

$$\frac{x\,(x+1)}{(1+2x+x^2)^2} = \frac{A\,x+B}{2x^2+2x+1} + \frac{C\,x+D}{(2x^2+2x+1)^2}$$

$$x^2 + x = (A\,x+B)\,(2x^2+2x+1) + (C\,x+D)$$

Dengan identiteit :

koeffisien x^3 : $0 = 2A$ dan $A = 0$

koeffisien x^2 : $1 = 2A + 2B$ dan $B = \frac{1}{2}$

koeffisien x : $1 = A + 2B + C$ dan $C = 0$

koeffisien x^0 : $0 = B + D$ dan $D = -\frac{1}{2}$

Maka :

$$\int \frac{x\,(1+x)\,dx}{(1+2x+2x)^2} = \tfrac{1}{2}\int \frac{dx}{2x^2+2x+1} = \tfrac{1}{2}\int \frac{dx}{(2x^2+2x+1)^2}$$

$$= \tfrac{1}{4}\int \frac{dx}{(x+\frac{1}{2})^2+\frac{1}{4}} - \frac{1}{8}\int \frac{dx}{\left[(x+\frac{1}{2})^2+\frac{1}{4}\right]^2}$$

$$= \tfrac{1}{2}\ \text{arc tg}\ (2x+1) - \frac{1}{8}\int \frac{dx}{\left[(x+\frac{1}{2})^2\,(\frac{1}{4})\right]^2}$$

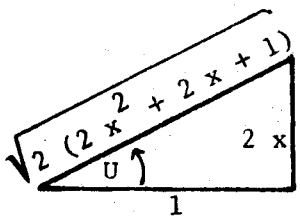
Dicari dulu bentuk :

$$\int \frac{dx}{\left[(x+\frac{1}{2})^2+(\frac{1}{4})\right]^2}\ ,\ \text{diambil substitusi : } x+\tfrac{1}{2} = \tfrac{1}{2}\ \text{tg}\ U$$

$$dx = \tfrac{1}{2}\sec^2 U\ dU$$

Setelah dimasukkan terdapat bentuk :

$$\int \frac{\frac{1}{2}\sec^2 U}{\frac{1}{16}\sec^4 U} = 8\int \cos^2 U\ dU = \frac{8}{2}\int (1+\cos 2U)\ dU$$



$$= \frac{8}{2}(U + \tfrac{1}{2}\sin 2U) + C = 4(\text{arc tg}(2x+1) + \frac{2x+1}{2(2x^2+2x+1)} + C$$

J a d i :

$$\int \frac{x(1+x)}{(1+2x+2x^2)^2}\,dx = \tfrac{1}{2}\,\text{arc tg}(2x+1) - \tfrac{1}{2}\,\text{arc tg}(2x+1) - \tfrac{1}{4}\left(\frac{2x-1}{2x^2+2x+1}\right) + C$$

$$\int \frac{x(1+x)\,dx}{(1+2x+2x^2)^2} = -\tfrac{1}{4}\,\frac{2x+1}{2x^2+2x+1} + C$$

16). Tentukan : $\int \frac{1}{x}\sqrt{\frac{x+4}{x+1}}\,dx$ (ujian tahun 1925).

Penyelesaian : lihat contoh soal didepan (hal 48 / No. 1)

17). Tentukan : $\int \frac{dx}{x\sqrt{x^2\sin^2\theta - 2x\cos\theta - 1}}$ (ujian th 1926).

Penyelesaian :

$$\int \frac{dx}{x\sqrt{x^2\sin^2\theta\ 2x\cos\theta - 1}} = \int \frac{dx}{x\sqrt{x^2(1-\cos^2\theta) - 2x\cos\theta - 1}}$$

$$= \int \frac{dx}{x\sqrt{x^2 - (x\cos\theta + 1)^2}}$$

$$= \int \frac{dx}{x\sqrt{[x(1-\cos\theta) - 1]\,[x(1+\cos\theta) + 1]}}$$

Misalkan :

$$x(1-\cos\theta) - 1 = [x(1+\cos\theta) + 1]\,U^2$$

$$(2\sin^2\tfrac{1}{2}\theta - 2U^2\cos^2\tfrac{1}{2}\theta)\,x = U^2 + 1$$

$$x = \frac{U^2 + 1}{2\,(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)}$$

$$dx = \tfrac{1}{2}\,\frac{2\,u\,(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta) + 2\,U\,(\cos^2 \tfrac{1}{2}\theta)(U^2 + 1)}{(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)^2}\,dU$$

$$= \frac{U\,(\sin^2 \tfrac{1}{2}\theta + \cos^2 \tfrac{1}{2}\theta)}{(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)^2}\,dU = \frac{U\ dU}{(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)^2}$$

Sekarang bentuk integral menjadi :

$$\int \frac{\dfrac{U\,dU}{(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)^2}}{\left[\dfrac{U^2 + 1}{2\,(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta}\right]\left[\dfrac{2\,U}{2\,(\sin^2 \tfrac{1}{2}\theta - U^2 \cos^2 \tfrac{1}{2}\theta)}\right]}$$

$$= 2\int \frac{dU}{U^2 + 1} = 2 \text{ arc tg } U + C.$$

$$= 2 \text{ arc tg } \left(\frac{x\,(1 - \cos\theta) - 1}{x\,(1 + \cos\theta) + 1}\right)^{\frac{1}{2}} + C.$$

J a d i :

$$\int \frac{dx}{x\sqrt{x^2 \sin^2\theta - 2x\cos\theta - 1}} = 2 \text{ arc tg } \left(\frac{x\,(1 - \cos\theta) - 1}{x\,(1 + \cos\theta) + 1}\right)^{\frac{1}{2}} + C$$

18). Tentukan : $\int \frac{\ln\,(1 + x^2)}{x^3}\,dx$ (ujian tahun 1927).

Penyelesaian :

$$\int \frac{\ln\,(1 + x^2)}{x^3}\,dx = -\tfrac{1}{2}\int \ln\,(1 + x^2)\,dx^{-2}$$

$$= -\tfrac{1}{2}\,\frac{\ln\,(1 + x^2)}{x^2} + \int \frac{x\,dx}{x^2\,(1 + x^2)}$$

$$= -\frac{\ln\,(1 + x^2)}{2\,x^2} + \int \frac{dx}{x} - \int \frac{x\,dx}{x^2 + 1}$$

$$= -\frac{\ln\,(1 + x^2)}{2\,x^2} - \ln\left|\frac{\sqrt{1 + x^2}}{x}\right| + C$$

Jadi :

$$\int \frac{\ln (1 + x^2)}{x^3}\, dx = - \ln \frac{1 + x^2}{x} - \frac{\ln (1 + x^2)}{2 x^2} + C$$

19). Integral : $\int \frac{1 - \cos x}{2 + \cos x} \sin x\, dx$ (ujian tahun 1928).

Penyelesaian :

$$\int \frac{1 - \cos x}{2 + \cos x} \sin x\, dx = - \int \frac{1 - \cos x}{2 + \cos x}\, d (\cos x)$$

Substitusi : $\frac{1 - \cos x}{2 + \cos x} = U^2$ dan $\cos x = \frac{1 - 2 U^2}{U^2 + 1}$

$$d (\cos x) = - \frac{6 U\, dU}{(U^2 + 1)^2}$$

M a k a :

$$\int \frac{1 - \cos x}{2 + \cos x} \sin x\, dx = - \int U \cdot \frac{- 6 U\, dU}{(U^2 + 1)^2} = 6 \int \frac{U^2 dU}{(U^2 + 1)^2}$$

$$= 6 \int \frac{dU}{U^2 + 1} - 6 \int \frac{dU}{(U^2 + 1)^2}$$

$$= 6 \text{ arc tg } U - 3 \text{ arc tg } U - 3 \frac{U}{u^2 + 1} + C.$$

$$= 3 \text{ arc tg} \sqrt{\frac{1 - \cos x}{2 + \cos x}} - \sqrt{(1 - \cos x)(2 + \cos x)} + C$$

20). Tentukan : $\int \frac{dx}{2 \sin^2 x - 3 \cos x}$ (ujian tahun 1929).

Penyelesaian :

Untuk menyelesaikan integral tersebut diatas, diambil substitusi :

$$U = \text{tg } \tfrac{1}{2} x \text{ , } x = 2 \text{ arc tg } U \text{ dan } dx = \frac{2\, dU}{1 + U^2}$$

$$\cos x = \frac{1 - U^2}{1 + U^2} \quad \text{dan} \quad \sin x = \frac{2U}{1 + U^2}$$

M a k a :

$$\int \frac{dx}{2 \sin^2 x - 3 \cos x} = \int \frac{\frac{2\,dU}{1 + U^2}}{2\left(\frac{2U}{1 + U^2}\right)^2 - 3\left(\frac{1 - U^2}{1 + U^2}\right)}$$

$$= 2\int \frac{(1 + U^2)\,dU}{3U^4 + 8U^2 - 3} = 2\int \frac{(1 + U^2)\,dU}{(3U^2 - 1)(U^2 + 3)}$$

$$= \frac{4}{5}\int \frac{dU}{3U^2 - 1} + \frac{2}{5}\int \frac{dU}{U^2 + 3}$$

$$= \frac{4}{15}\sqrt{3}\ \text{arc tg}\ U\sqrt{3} + \frac{2}{15}\sqrt{3}\ \text{arc tg}\ \frac{U}{\sqrt{3}} + C.$$

Jadi bentuk :

$$\int \frac{dx}{2 \sin^2 x - 3 \cos x} = \frac{4}{15}\sqrt{3}\ \text{arc tg}\ \sqrt{3}\,(\text{tg}\ \tfrac{x}{2}) + \frac{2}{15}\sqrt{3}\ \text{arc tg}\ \tfrac{1}{3}\sqrt{3}\,(\text{tg}\ \tfrac{x}{2}) + C.$$

21). Hitung integral : a) $\int \frac{dx}{(1 - x^2)^{3/2}}$ dan b) $\int \frac{\text{arc sin } x}{(1 - x^2)^{3/2}}\,dx$

(ujian th. 1930).

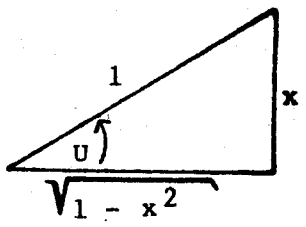
Penyelesaian :

a). $\int \frac{dx}{(1 - x^2)^{3/2}}$, diambil substitusi :

$$x = \sin U \text{ dan } dx = \cos U\,dU$$

Maka integral sekarang , berbentuk :

$$\int \frac{\cos U\,dU}{\cos^3 U} = \int \sec^2 U\,dU = \text{tg}\ U + C$$



Jadi : $\int \frac{dx}{(1 - x^2)^{3/2}} = \frac{x}{\sqrt{1 - x^2}} + C.$

b). $\int \frac{\text{arc sin } x}{(1 - x^2)^{3/2}} dx$, untuk menyelesaikan integral ini diambil substitusi :

$x = \sin U$ dan $dx = \cos U \, du$

Maka integral menjadi :

$$\int \frac{U \cos U \, dU}{\cos^3 U} = \int U \sec^2 U \, dU = \int U \, d(\text{tg } U) = U \text{ tg } U - \int \text{tg } U \, dU$$

$$= U \text{ tg } U + \ln \cos U + C.$$

J a d i :

$$\int \frac{\text{arc sin } x}{(1 - x^2)^{3/2}} dx = \frac{x}{\sqrt{1 - x^2}} \text{ arc sin } x + \ln \sqrt{1 - x^2} + C.$$

$$= \frac{x}{\sqrt{1 - x^2}} \text{ arc sin } x + \tfrac{1}{2} \ln \left| 1 - x^2 \right| + C.$$

22). Tentukan : $\int \frac{dx}{\cos^4 x + \sin^4 x}$ (ujian tahun 1924/1930)

Penyelesaian :

$$\cos^4 x + \sin^4 x = (\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x$$

$$= 1 - \tfrac{1}{2} \sin^2 2x = \tfrac{1}{2} (2 - \sin^2 2x)$$

$$= \tfrac{1}{4} (4 - 1 - \cos 4x) = \tfrac{1}{4} (3 - \cos 4x)$$

Maka integral :

$$\int \frac{dx}{\cos^4 x + \sin^4 x} = \int \frac{4 \, dx}{3 - \cos 4x} = \int \frac{d(4x)}{3 - \cos 4x}$$

Digunakan substitusi : $4x = 2 \text{ arc tg } U$ dan $d(4x) = \frac{2 \, dU}{1 + U^2}$

$$\cos 4x = \frac{1 - U^2}{1 + U^2}$$

Bentuk integral menjadi :

$$\int \frac{1}{3 - \frac{1 - U^2}{1 + U^2}} \cdot \frac{2\,dU}{1 + U^2} = 2 \int \frac{dU}{4U^2 + 2} = \tfrac{1}{2} \int \frac{dU}{U^2 + \tfrac{1}{2}}$$

$$= \tfrac{1}{2} \sqrt{2} \text{ arc tg } U \sqrt{2} + C.$$

Jadi : $\int \frac{dx}{\cos^4 x + \sin^4 x} = \tfrac{1}{2} \sqrt{2} \text{ arc tg } \sqrt{2} (\text{tg } 2x) + C$

23). Hitung integral : $\int \frac{\sqrt{x^2 + 2x - 1}}{x} dx$ (ujian tahun 1930)

Penyelesaian :

Misalkan : $x^2 + 2x - 1 = (U - x)^2$, $x^2 + 2x - 1 =$

$= U^2 - 2xU + x^2$

$x = \frac{U^2 + 1}{2(U + 1)}$, $dx = \tfrac{1}{2} \frac{2U(U + 1) - (U^2 + 1)}{(U + 1)^2} dU$

Sedang : $\sqrt{x^2 + 2x - 1} = U - \frac{U^2 + 1}{2(U + 1)} = \frac{U^2 + 2U - 1}{2(U + 1)}$

M a k a :

$$\int \frac{\sqrt{x^2 + 2x - 1}}{x} dx = \tfrac{1}{2} \int \frac{(U^2 + 2U - 1)^2}{(U + 1)^2 (U^2 + 1)} dU$$

$$= \tfrac{1}{2} \int dU + \tfrac{1}{2} \int \frac{2U^2 - 6U}{(U + 1)^2 (U^2 + 1)} dU$$

$$= \tfrac{1}{2} U + \int \frac{dU}{U + 1} + \int \frac{dU}{[illegible]^2} - 2 \int \frac{dU}{U^2 + 1}$$

$$= \tfrac{1}{2} U + \ln (U + 1) - \frac{1}{U + 1} - 2 \text{ arc tg } U + C$$

J a d i :

$$\int \frac{x^2 + 2x - 1}{x}\,dx = \tfrac{1}{2}\left(\sqrt{x^2 + 2x - 1} + x\right)$$

$$+ \ln\left|\sqrt{x^2 + 2x - 1} + x + 1\right| - \frac{1}{\sqrt{x^2 + 2x - 1} + x + 1}$$

$$- 2 \operatorname{arc\,tg}\left(\sqrt{x^2 + 2x - 1} + x\right) + C.$$

24). Hitung integral :

a). $\int \frac{\operatorname{arc\,tg} x}{x^3}\,dx$ dan b). $\int \frac{1 - x^5}{(1 - x)(1 + x^4)}\,dx$ **(1931).**

Penyelesaian :

a). $\int \frac{\operatorname{arc\,tg} x}{x^3}\,dx = -\tfrac{1}{2}\int \operatorname{arc\,tg} x\, d\left(\frac{1}{x^2}\right)$

$$= -\frac{\operatorname{arc\,tg} x}{2x^2} + \tfrac{1}{2}\int \frac{dx}{x^2(1 + x^2)}$$

$$= -\frac{\operatorname{arc\,tg} x}{2x^2} + \tfrac{1}{2}\int \frac{dx}{x^2} - \tfrac{1}{2}\int \frac{dx}{x^2 + 1}$$

$$= -\frac{\operatorname{arc\,tg} x}{2x^2} - \frac{1}{2x} - \tfrac{1}{2}\operatorname{arc\,tg} x + C.$$

b). $\int \frac{1 - x^5}{(1 - x)(1 + x^4)}\,dx = \int \frac{1 + x + x^2 + x^3 + x^4}{1 + x^4}\,dx$

$$= \int \left(1 + \frac{x + x^2 + x^3}{x^4 + 1}\right) dx$$

$$= \int dx + \int \frac{x^3}{x^4 + 1}\,dx + \int \frac{x^2 + x}{x^4 + 1}\,dx$$

$$= x + \tfrac{1}{4}\ln(x^4 + 1) + \int \frac{x^2 + x}{x^4 + 1}\,dx$$

Pandang dulu bentuk :

$$\frac{x^2 + x}{x^4 + 1} = \frac{x^2 + x}{(x^2 + 1)^2 - 2x^2} = \frac{x^2 \; 1 \; x}{(x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)}$$

$$= \frac{A\,x + B}{x^2 + x\sqrt{2} + 1} + \frac{C\,x + D}{x^2 - x\sqrt{2} + 1}$$

$$x^2 + x = (a\,x + B)(x^2 - x\sqrt{2} + 1) + (C\,x + D)(x^2 + x\sqrt{2} + 1)$$

Dengan identiteit , terdapat :

koeffisien x^3 : $0 = \quad A + C$

koeffisien x^2 : $1 = -A\sqrt{2} + B + C\sqrt{2} + D$

koeffisien x^1 : $1 = \quad A - B\sqrt{2} + C + D\sqrt{2}$

koeffisien x^o : $0 = \quad B + D$

Setelah dihitung terdapat :

$A = -\frac{1}{4}\sqrt{2}$, $B = -\frac{1}{4}\sqrt{2}$, $C = \frac{1}{4}\sqrt{2}$ dan $D = \frac{1}{4}\sqrt{2}$.

M a k a :

$$\int \frac{(x^2 + x)\,dx}{x^4 + 1} = \tfrac{1}{4}\sqrt{2}\int \frac{(x + 1)\,dx}{x^2 - x\sqrt{2} + 1} - \tfrac{1}{4}\sqrt{2}\int \frac{(x + 1)\,dx}{x^2 + x\sqrt{2} + 1}$$

$$= \tfrac{1}{4}\sqrt{2}\int \frac{x\,dx}{x^2 - x\sqrt{2} + 1} + \tfrac{1}{4}\sqrt{2}\int \frac{dx}{(x - \tfrac{1}{2}\sqrt{2})^2 + \tfrac{1}{2}}$$

$$- \tfrac{1}{4}\sqrt{2}\int \frac{x\,dx}{x^2 + x\sqrt{2} + 1} - \tfrac{1}{4}\sqrt{2}\int \frac{dx}{(x + \tfrac{1}{2}\sqrt{2})^2 + \tfrac{1}{2}}$$

$$= \frac{1}{8}\sqrt{2}\int \frac{(2x - \sqrt{2})\,dx}{x^2 - x\sqrt{2} + 1} + \left(\frac{1 + \sqrt{2}}{4}\right)\sqrt{2}\ \text{arc tg}$$

$$(x\sqrt{2} - 1) - \frac{1}{8}\sqrt{2}\ \frac{(x + \sqrt{2})\,dx}{x^2 + x\sqrt{2} + 1} - \left(\frac{1 + \sqrt{2}}{4}\right)\sqrt{2}\ \text{arc tg}$$

$$(x\sqrt{2} + 1)$$

$$= \frac{1}{8}\sqrt{2}\ln\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1} + \frac{2+\sqrt{2}}{4}\operatorname{arc\,tg}(x\sqrt{2} - 1)$$

$$- \frac{2+\sqrt{2}}{4}\operatorname{arc\,tg}(x\sqrt{2} + 1) + C.$$

Hasil seluruhnya dari :

$$\int \frac{1 - x^5}{(1-x)(1+x^4)}\,dx = x + \tfrac{1}{4}\ln(x^4 + 1) + \frac{1}{8}\sqrt{2}\ln$$

$$\left|\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right| + \frac{2+\sqrt{2}}{4}\operatorname{arc\,tg}(x\sqrt{2} - 1) - \frac{2+\sqrt{2}}{4}\operatorname{arc\,tg}$$

$$(x\sqrt{2} + 1) + C.$$

25). Tentukan : $\int \frac{x^4 - 1}{x^4 + 1}\,dx$ (ujian th 1931).

Penyelesaian :

Disini pangkat pembilang dan penyebut sama , maka dilakukan pembagian lebih dulu.

$$\int \frac{x^4 - 1}{x^4 + 1}\,dx = \int \frac{x^4 + 1 - 2}{x^4 + 1}\,dx = \int dx - 2\int \frac{dx}{x^4 + 1}$$

$$= x - 2\int \frac{dx}{(x^2 + x\sqrt{2} + 1)(x^2 - x\sqrt{2} + 1)} \quad \text{(lihat soal 24)}$$

$$= x - \int \frac{\frac{1}{2}x\ 2 + 1}{x^2 + x\sqrt{2} + 1}\,dx + \int \frac{\frac{1}{2}x\sqrt{2} - 1}{x^2 - x\sqrt{2} + 1}\,dx$$

$$= x - \tfrac{1}{4}\sqrt{2}\int \frac{(2x + \sqrt{2})\,dx}{x^2 + x\sqrt{2} + 1} - \tfrac{1}{2}\int \frac{dx}{(x + \frac{1}{2}\sqrt{2})^2 + \frac{1}{2}}$$

$$+ \tfrac{1}{4}\sqrt{2}\int \frac{(2x - 2)\,dx}{x^2 - x\ 2 + 1} - \tfrac{1}{2}\int \frac{dx}{(x - \frac{1}{2}\sqrt{2})^2 + \frac{1}{2}}$$

$$= x + \tfrac{1}{4}\sqrt{2}\ln\left|\frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1}\right| - \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(x\sqrt{2} + 1)$$

$$- \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(x\sqrt{2} - 1) + C.$$

J a d i :

$$\int \frac{x^4 - 1}{x^4 + 1}\, dx = x + \tfrac{1}{4}\sqrt{2}\, \ln \left| \frac{x^2 - x\sqrt{2} + 1}{x^2 + x\sqrt{2} + 1} \right| -$$

$$\tfrac{1}{2}\sqrt{2}\ \text{arc tg}\ (x\sqrt{2} + 1) - \tfrac{1}{2}\sqrt{2}\ \text{arc tg}\ (x\sqrt{2} - 1) + C.$$

26). Hitung integral :

a). $\int \frac{dx}{x^2 (1 + x^2)^3}$; b). $\int \sin x \sin 2x \sin 3x\, dx$

(ujian tahun 1931)

Penyelesaian :

a). $\int \frac{dx}{x (1 + x^2)^3}$. Diambil substitusi $x = \text{tg}U$, $dx = \sec^2 U\, dU$

Maka :

$$\int \frac{dx}{x (1 + x^2)^3} = \int \frac{\sec^2 U\, dU}{\text{tg}\, U \sec^6 U} = \int \frac{\cos^5 U}{\sin U}\, dU$$

$$= \int \frac{(1 - \sin^2 U)^2}{\sin U}\, d(\sin U)$$

$$= \int \frac{1 - 2\sin^2 U + \sin^3 U}{\sin U}\, d(\sin U)$$

$$= \ln \sin U - \sin^2 U + \frac{1}{3}\sin^3 U + C.$$

$$= \ln \left| \frac{x}{\sqrt{x^2 + 1}} \right| - \frac{x^2}{x^2 + 1} + \frac{1}{3} \cdot \frac{x^3}{(x^2 + 1)^{3/2}} + C.$$

b). $\int \sin x \sin 2x \sin 3x\, dx = \int \sin 2x \sin 3x \sin x\, dx$

$$= \int \sin 2x\ \tfrac{1}{2}(\cos 2x - \cos 4x)\ dx$$

$$= \tfrac{1}{2}\int \sin 2x \cos 2x\, dx - \tfrac{1}{2}\int \cos 4x \sin 2x\, dx$$

$$= \tfrac{1}{4}\int \sin 2x\, d(\sin 2x) + \tfrac{1}{4}\int (2\cos^2 2x - 1)\, d\cos 2x$$

$$= \frac{1}{8}\sin^2 2x + \frac{1}{6}\cos^3 2x - \tfrac{1}{4}\cos 2x + C.$$

27). Hitung integral :

a). $\int \frac{dx}{\sqrt{(1 + x^2)^3}}$ dan b). $\int \frac{\text{arc tg } x}{\sqrt{(1 + x^2)^3}} dx$ **(ujian th. 1932)**

Penyelesaian :

a). $\int \frac{dx}{\sqrt{(1 + x^2)^3}}$. Diambil substitusi :

$x = \text{tg } U$ dan $dx = \sec^2 U \, dU$

Maka bentuk integral menjadi :

$$= \int \frac{\sec^2 U \, dU}{\sec^3 U} = \int \cos U \, dU = \sin U + C$$

J a d i :

$$\int \frac{dx}{\sqrt{(1 + x^2)^3}} = \frac{x}{\sqrt{x^2 + 1}} + C.$$

b). $\int \frac{\text{arc tg } x}{\sqrt{(1 + x^2)^3}} dx$. Untuk integral ini, diambil :

$x = \text{tg } U$, $dx = \sec^2 U \, dU$ dan arc tg x = U

Maka integral sekarang menjadi :

$$= \int \frac{U \sec^2 U \, dU}{\sec^3 U} = \int U \cos U \, dU = \int U \, d \, (\sin U)$$

$$= U \sin U \cos U + C.$$

Jadi :

$$\int \frac{\text{arc tg } x}{\sqrt{(1 + x^2)^3}} dx = \frac{x}{\sqrt{1 + x^2}} \text{ arc tg } x + \frac{1}{\sqrt{1 + x^2}} + C.$$

28). Tentukan : $\int \frac{\ln x}{\sqrt{1 + x}} dx$ **(ujian tahun 1932)**

Penyelesaian :

Digunakan integrasi persiil :

$$\int \frac{\ln x}{\sqrt{1+x}}\,dx = 2\int \ln x\, d\left(\sqrt{1+x}\right)$$

$$= 2\sqrt{1+x}\ln x - 2\int \frac{\sqrt{x+1}}{x}\,dx$$

Pandang dulu bentuk : $\int \frac{\sqrt{x+1}}{x}\,dx$, diambil substitusi

$x + 1 = U^2$, $dx = 2\,U\,dU$. Maka integral setelah ada substitusi adalah :

$$= \int \frac{U.\ 2\cdot U\,dU}{U^2 - 1} = 2\int \frac{U^2\,dU}{U^2 - 1} = 2\int dU + 2\int \frac{dU}{U^2 - 1}$$

$$= 2\,U + \ln\left|\frac{U-1}{U+1}\right| + C.$$

J a d i :

$$\int \frac{\ln x}{\sqrt{x+1}}\,dx = 2\sqrt{x+1}\ln x - 4\sqrt{x+1} - 2\ln$$

$$\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

29). Tentukan : $\int \frac{9x^3 - 15x^2 - 6x + 12}{x^5 - 2x^4 + x^3 + 8x^2 - 16x + 8}\,dx$

(ujian tahun 1933)

Penyelesaian :

Bentuk : $x^5 - 2x^4 + x^3 + 8x^2 - 16x + 8 = (x-1)^2\,(x+2)$

$(x^2 - 2x + 4)$

$$\frac{9x^3 - 15x^2 - 6x + 12}{x^5 - 2x^4 + x^3 + 8x^2 - 16x + 8} = \frac{A}{x+2} + \frac{B}{x-1} +$$

$$+ \frac{C}{(x-1)^2} + \frac{dx + E}{x^2 - 2x + 4}$$

$(x^3 - 15\,x^2 - 6\,x + 12 = A\,(x-1)^2\,(x^2 - 2\,x + 4) + B\,(x+2)$

$(x-1)\,(x^2 - 2\,x + 4) + C\,(x+2)\,(x^2 - 2\,x + 4) + (Dx + E)$

$(x+2)\,(x-1)^2$

Untuk $x = -2$, $-108 = A\,(-3)^2\,(12)$ dan $A = -1$

$x = 1$, $0 = C\,(3)\,(3)$ dan $C = 0$

Dimasukkan harga-harga ini, kemudian disederhanakan :

$(9\,x^2 - 6\,x - 12)\,(x-1) + (x-1)^2\,(x^2 - 2\,x + 4) = (x-1)$

$\left[B\,(x+2)\,.\,(x^2 - 2\,x + 4) + (D\,x + E)\,(x+2)\,(x-1)\right]$

$x^3 + 6\,x^2 - 16 = (x+2)\left[B\,(x^2 - 2\,x + 4) + (D\,x + E)\,(x-1)\right]$

$x^2 + 4\,x - 8 = B\,(x^2 - 2\,x + 4) + (Dx + E)\,(x-1)$

Untuk $x = 1$, $-3 = 3\,B$, $B = -1$

Dengan identiteit , terdapat :

$B + D = 1 \longrightarrow D = 2$; $4\,B - E = -8 \longrightarrow E = 4$, maka :

$$\int \frac{9\,x^3 - 15\,x^2 - 6\,x + 12}{x^5 - 2\,x^4 + x^3 + 8\,x^2 - 16\,x + 8}\,dx = -\int \frac{dx}{x+2} - \int \frac{dx}{x-1} + \int \frac{2\,x + 4}{x^2 - 2\,x + 4}\,dx.$$

$$= -\ln(x+2) - \ln(x-1) + \ln(x^2 - 2x + 4) + 6 \int \frac{dx}{(x-1)^2 + (\quad 3)^2}$$

$$= \ln\left|\frac{x^2 - 2x - 4}{x^2 + x - 2}\right| + 2\sqrt{3}\ \text{arc tg}\ \frac{x-1}{\sqrt{3}} + C.$$

30). Tentukan : $\int \frac{\sin x \cos^2 x \, dx}{\cos^2 x + a \cos x + a^2}$ (ujian tahun 1934)

Penyelesaian :

$$\int \frac{\sin x \cos^2 x \, dx}{\cos^2 x + a \cos x + a^2} =$$

$$-\int \frac{(\cos^2 x + a \cos x + a^2) - (a \cos x + a^2)}{\cos^2 x + a \cos x + a^2} \, d \cos x$$

$$= -\int d (\cos x) + \int \frac{a (\cos x + a)}{\cos^2 + x + a \cos x + a^2} \, dx$$

$$= - \cos x + \tfrac{1}{2} a \ln (\cos^2 x + a \cos x + a^2)$$

$$+ \tfrac{1}{2} a^2 \int \frac{d (\cos x)}{(\cos x + \tfrac{1}{2} a)^2 + (\tfrac{1}{2} a \sqrt{3})^2}$$

$$= - \cos x + \tfrac{1}{2} a \ln (\cos^2 x + a \cos x + a^2)$$

$$+ \tfrac{1}{2} a \sqrt{3} \text{ arc tg } \left(\frac{2 \cos x + a}{a \sqrt{3}}\right) + C.$$

31). Tentukan : $\int (\text{arc sin } x)^3 \, x \, dx$ (ujian th. 1934)

Penyelesaian :

Untuk menyelesaikan integral diatas ini , diambil substitusi arc sin x = U , x = sin U dan dx = cos U dU . Maka :

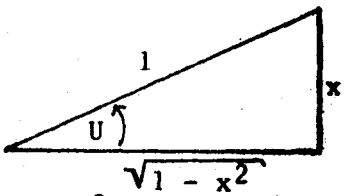
$$\int (\text{arc sin } x)^3 \, x \, dx = \int U^3 \sin U \cos U \, dU = \tfrac{1}{2} \int U^3 \sin 2 U \, dU$$

$$= - \tfrac{1}{4} \int U^3 \, d (\cos 2 U) = - \tfrac{1}{4} U^3 \cos 2 U + \frac{3}{4} \int U^2 \cos 2 U \, dU$$

$$= - \tfrac{1}{4} U^3 \cos 2 U + \frac{3}{8} U^2 \sin 2 U + \frac{3}{8} U \cos 2 U$$

$$- \frac{3}{16} \sin 2 U + C.$$

J a d i :



$$\int (\text{arc sin } x)^3 \, x \, dx = \tfrac{1}{4} (\text{arc sin } x)^3 (1 - 2x^2) + \frac{3}{8}\left[2x\sqrt{1 - x^2} (\text{arc sin } x)^2 - x\sqrt{1 - x^2}\right] + \frac{3}{8}(2x^2 - 1) \text{ arc sin } x + C.$$

32). Tentukan : $\int \text{tg } x \text{ tg } (x - a) \, dx$ (ujian tahun 1935)

Penyelesaian :

Kita ingat pada rumus : $\text{tg } (A - B) = \dfrac{\text{tg } A - \text{tg } B}{1 + \text{tg } A \text{ tg } B}$

atau $\text{tg } A \text{ tg } B = \dfrac{\text{tg } A - \text{tg } B}{\text{tg } (A - B)} - 1$, maka :

$$\text{tg } x \text{ tg } (x - a) = \frac{\text{tg } x - \text{tg } (x - a)}{\text{tg } (x - x + a)} - 1$$

J a d i :

$$\int \text{tg } x \text{ tg } (x - a) \, dx = \frac{1}{\text{tg } a} \int \left[\text{tg } x - \text{tg } (x - a) \right] dx - \int dx$$

$$= \text{ctg } a \left[\ln \sec x + \ln \cos (x - a) \right] - x + C$$

$$= \text{ctg } a \left| \ln \frac{\cos (x - a)}{\cos x} \right| - x + C$$

33). Tentukan : $\int \dfrac{x}{(a^2 + x^2) \ln (a^2 + x^2)} \, dx$ (ujian th. 1935)

Penyelesaian :

$$\int \frac{x}{(a^2 + x^2) (\ln (a^2 + x^2)} \, dx = \tfrac{1}{2} \int \frac{d (a^2 + x^2)}{(a^2 + x^2) \ln (a^2 + x^2)}$$

$$= \tfrac{1}{2} \int \frac{d \ln (a^2 + x^2)}{\ln (a^2 + x^2)} = \tfrac{1}{2} \ln \left[\ln (a^2 + x^2) \right] + C.$$

34). Tentukan : $\int \frac{dx}{(1 + x^2)\ (x + \sqrt{1 + x^2})}$ (ujian th. 1936).

Penyelesaian :

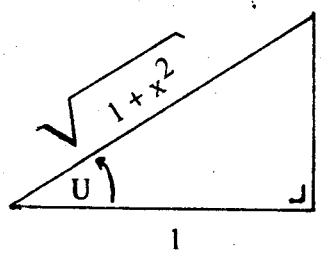
Untuk menyelesaikan integral tersebut , diambil substitusi :

$x = tg\ U$, $dx = \sec^2 U\ dU$; sedang $1 + x^2 = 1 + tg^2\ U = \sec^2 U$

M a k a :

$$\int \frac{dx}{(1 + x^2)\ (x + \sqrt{1 + x^2})} = \int \frac{\sec^2 U\ dU}{\sec^2 U\ (tg\ U + \sec U)}$$

$$= \int \frac{\cos U\ dU}{\sin U + 1} = \ln (\sin U + 1) + C.$$



$$= \ln \left(\frac{x}{\sqrt{1 + x^2}} + 1\right) + C.$$

$$= \ln (\sqrt{1 + x^2} + x) - \tfrac{1}{2} \ln (1 + x^2) + C$$

35). Tentukan : $\int \frac{a + \sqrt{a^2 - x^2}}{x^2}\ dx$ (ujian tahun 1936)

Penyelesaian :

Untuk menyelesaikan integral ini , diambil substitusi :

$$a^2 - x^2 = (a - x)^2\ U^2\ ,\ a + x = (a - x)\ U^2,\ x = \frac{a(U^2 - 1)}{U^2 + 1}$$

$$dx = a\ \frac{2\ U\ (U^2 + 1) - 2\ U\ (U^2 - 1)}{(U^2 + 1)^2}\ dU = \frac{4\ U\ dU}{(U^2 + 1)^2}$$

$$a^2 - x^2 - a = a - \frac{a\ (U^2 - 1)^2}{U^2 + 1}\ U^2 = a^2\ \left(\frac{U^2 + 1 - U^2 + 1}{U^2 + 1}\right)^2\ U^2$$

$$= \left[\ \frac{2\ aU}{U^2 + 1}\ \right]^2$$

sekarang integral :

$$\int \frac{a + \sqrt{a^2 - x^2}}{x^2}\, dx = \int \frac{a + \frac{2aU}{U^2+1}}{a^2 \left(\frac{U^2-1}{U^2+1}\right)^2} \cdot \frac{4aU\, dU}{(U^2+1)^2}$$

$$= \int \frac{4U(U+1)^2\, dU}{(U^2+1)(U+1)^2(U-1)^2} = \int \frac{4U\, dU}{(U-1)^2(U^2+1)}$$

Pandang dulu bentuk :

$$\frac{4U}{(U-1)^2(U^2+1)} = \frac{A}{U-1} + \frac{B}{(U-1)^2} + \frac{CU+D}{U^2+1}$$

$$4U = A(U-1)(U^2+1) + B(U^2+1) + (CU+D)(U-1)^2$$

Untuk $U = 1$, $4 = 2B$ dan $B = 2$

Dengan identiteit :

koeffisien U^3 : $0 = A + C$

koeffisien U^2 : $0 = -A + B - 2C + D$

koeffisien U : $4 = +A - C - 2D$

koeffisien U^0 : $0 = -A + B - D$

Setelah dihitung terdapat : $A = 0$, $C = 0$ dan $D = -2$.

M a k a :

$$\int \frac{4U\, dU}{(U-1)^2(U^2+1)} = 2\int \frac{dU}{(U-1)^2} - 2\int \frac{dU}{U^2+1}$$

$$= -\frac{2}{U-1} - 2 \text{ arc tg } U + C.$$

J a d i :

$$\int \frac{a + \sqrt{a^2 - x^2}}{x^2}\, dx = -\frac{2\sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}} - 2 \text{ arc tg } \left(\frac{a + x}{a - x}\right)^{\frac{1}{2}} + C.$$

36). Tentukan : $\int \frac{x}{\sqrt{1 - x^2}} \ln \left(\frac{1 + x}{1 - x}\right) dx$ (ujian tahun 1936)

Penyelesaian :

$$\int \frac{x}{\sqrt{1 - x^2}} \ln \left(\frac{1 + x}{1 - x}\right) dx = -\int \ln \left(\frac{1 + x}{1 - x}\right) d\left(\sqrt{1 - x^2}\right)$$

$$= -\sqrt{1 - x^2} \ln \left(\frac{1 + x}{1 - x}\right) + \int \sqrt{1 - x^2}\, d\left(\ln \left(\frac{1 + x}{1 - x}\right)\right)$$

$$= -\sqrt{1 - x^2} \ln \left(\frac{1 + x}{1 - x}\right) + \int \sqrt{1 - x^2} \cdot \frac{2}{1 - x^2}\, dx$$

$$= -\sqrt{1 - x^2} \ln \left(\frac{1 + x}{1 - x}\right) + 2\int \frac{dx}{\sqrt{1 - x^2}}$$

$$= -\sqrt{1 - x^2} \ln \left(\frac{1 + x}{1 - x}\right) + 2 \text{ arc sin } x + C.$$

J a d i :

$$\int \frac{x}{\sqrt{1 - x^2}} \ln \left(\frac{1 + x}{1 - x}\right) dx = -\sqrt{1 - x^2} \ln \left(\frac{1 + x}{1 - x}\right) + 2 \text{ arc sin } x + C.$$

37). Tentukan integral :

$$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 + x^2}}, \quad (a^2 > b^2)$$ (ujian th. 1937)

Penyelesaian :

Untuk menyelesaikan integral tersebut , dimisalkan :

$x = a \text{ tg } U$ dan $dx = a \sec^2 U\, dU$, maka :

$$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 + x^2}} = \int \frac{a \sec^2 U}{(b^2 + a^2 + tg^2 U)\, a \sec U}\, dU$$

$$= \int \frac{d(\sin U)}{b^2 \cos^2 U + a^2 \sin^2 U} = \int \frac{d(\sin U)}{b^2 + (a^2 - b^2)\sin^2 U}$$

$$= \frac{1}{b\sqrt{a^2 - b^2}} \text{ arc tg } \frac{\sqrt{a^2 - b^2}\sin U}{b} + C.$$

J a d i :

$$\int \frac{dx}{(b^2 + x^2)\sqrt{a^2 + x^2}} = \frac{1}{b\sqrt{a^2 - b^2}} \text{ arc tg } \left(\frac{x\sqrt{a^2 - b^2}}{b\ a^2 + x^2} \right) + C$$

38). Tentukan : $\int \frac{\text{arc tg } x}{\sqrt{(x^2 + 1)^5}}\, dx$ (ujian tahun 1938).

Penyelesaian :

Andaikan : $x = tg\ U$, $dx = \sec^2 U\, dU$ dan arc tg $x = U$.

$$\int \frac{\text{arc tg } x}{\sqrt{(x^2 + 1)^5}}\, dx = \int \frac{U \sec^2 U\, dU}{\sec^5 U} = \int U \cos^3 U\, dU$$

$$= \int U (1 - \sin^2 U)\, d(\sin U) = \int u\, d(\sin U - \tfrac{1}{3}\sin^3 U)$$

$$= U(\sin U - \tfrac{1}{3}\sin^3 U) - \int (\sin U - \tfrac{1}{3}\sin^3 U)\, dU$$

$$= U(\sin U - \tfrac{1}{3}\sin^3 U) + \cos U + \tfrac{1}{3}\int \sin^2 U \sin U\, dU$$

$$= U \sin U + \cos U - \tfrac{1}{3} U \sin^3 U - \tfrac{1}{3}\int (1 - \cos^2 U)\, d \cos U$$

$$= U \sin U + \tfrac{2}{3}\cos U - \tfrac{1}{3} U \sin^3 U + \tfrac{1}{9}\cos^3 U + C$$

J a d i :

$$\int \frac{\text{arc tg } x}{\sqrt{(x^2 + 1)^5}}\, dx = \frac{x \text{ arc tg } x}{\sqrt{x^2 + 1}} + \frac{2}{3\sqrt{x^2 + 1}} - \frac{x^3 \text{ arc tg } x}{3\sqrt{(x^2 + 1)^3}}$$

$$+ \frac{1}{9\sqrt{(x^2 + 1)^3}} + C.$$

$$= \frac{3x \text{ arc tg } x + 2}{3\sqrt{x^2 + 1}} + \frac{1 - 3x^3 \text{ arc tg } x}{9\sqrt{(x^2 + 1)^3}} + C.$$

39). Tentukan : $\int \frac{dx}{\sin \frac{1}{2} x \sqrt{1 + \cos x + \sin x}}$ (ujian th. 1939).

Penyelesaian :

Misalkan : $\text{tg } \frac{x}{2} = U$, $x = 2 \text{ arc tg } U$, $dx = \frac{2\,dU}{1 + U^2}$

$$\cos x = \frac{1 - U^2}{1 + U^2} , \quad \sin x = \frac{2U}{1 + U^2} , \quad \sin \tfrac{1}{2} x = \frac{U}{\sqrt{1 + U^2}}$$

M a k a :

$$\int \frac{dx}{\sin \frac{1}{2} x \quad 1 + \cos x + \sin x} = \int \frac{\frac{2\,dU}{1 + U^2}}{\frac{U}{\sqrt{1 + U^2}} \sqrt{1 + \frac{1 - U^2}{1 + U^2} + \frac{2U}{1 + U^2}}}$$

$$= \sqrt{2} \int \frac{dU}{U\sqrt{U + 1}}$$

Misalkan : $1 + U = t^2$, $U = t^2 - 1$ dan $dU = 2t\,dt$.

M a k a :

$$\sqrt{2} \int \frac{dU}{U\sqrt{U + 1}} = 2\sqrt{2} \int \frac{t\,dt}{(t^2 - 1)\,t} = 2\sqrt{2} \int \frac{dt}{t^2 - 1}$$

$$= \sqrt{2} \ln \left| \frac{t - 1}{t + 1} \right| + C = \sqrt{2} \ln \left| \frac{\sqrt{U + 1} - 1}{\sqrt{U + 1} + 1} \right| + C$$

J a d i :

$$\int \frac{dx}{\sin \frac{1}{2} x \sqrt{1 + \cos x + \sin x}} = 2 \ln \left| \frac{\sqrt{\operatorname{tg} \frac{x}{2} + 1} - 1}{\sqrt{\operatorname{tg} \frac{x}{2} + 1} + 1} \right| + C$$

40). Tentukan : $\int \frac{1}{\sqrt{x}} \operatorname{arc\ tg}\ x\ dx$

Penyelesaian :

$$\int \frac{1}{\sqrt{x}} \operatorname{arc\ tg}\ x\ dx = 2 \int \operatorname{arc\ tg}\ x\ d\ (X\sqrt{2})$$

$$= 2 \sqrt{x} \operatorname{arc\ tg}\ x - 2 \int \frac{\sqrt{x}}{1 + x^2}\ dx$$

Pandang dulu integral :

$\int \frac{\sqrt{x}}{1 + x^2}\ dx$. Misalkan $x = U^2$ dan $dx = 2\ U\ dU$, maka integral menjadi :

$$2 \int \frac{U^2\ dU}{U^4 + 1} = 2 \int \frac{U^2\ dU}{(U^2 - U\sqrt{2} + 1)\ (U^2 + U\sqrt{2} + 1)}$$

Pandang dulu bentuk :

$$\frac{U^2}{(U^2 - U\sqrt{2} + 1)\ (U^2 + U\sqrt{2} + 1)} = \frac{AU + B}{U^2 - U\sqrt{2} + 1} +$$

$$+ \frac{C\ U + D}{U^2 + U\sqrt{2} + 1}$$

$$U^2 + (AU + B)\ (U^2 + U\sqrt{2} + 1) + (CU + D)\ (U^2 - U\sqrt{2} + 1)$$

Dengan identiteit :

koeffisien U^3 : $A + C = 0$ dan $A = -C$

koeffisien U^2 : $A\sqrt{2} + B - C\sqrt{2} + D = 1 \longrightarrow 2\ A\sqrt{2} + B + D = 1$

koeffisien U^1 : $A + B\ \ 2 + C - D\ \ 2 = 0$

koeffisien U^0 : B + D = 0 , maka 2 A $\sqrt{2}$ = 1 dan A = $\frac{1}{4}\sqrt{2}$,

C = $\frac{1}{4}\sqrt{2}$, sedang B = D = 0

$$2\int\frac{U^2\,dU}{U^4+1} = \tfrac{1}{4}\sqrt{2}\int\frac{2U\,dU}{U^2-U\ 2+1} - \tfrac{1}{4}\sqrt{2}\int\frac{2U\,dU}{U^2+U\ 2+1}$$

$$= \tfrac{1}{4}\sqrt{2}\int\frac{(2U-\sqrt{2})\,dU}{U^2-U\sqrt{2}+1} + \tfrac{1}{2}\int\frac{dU}{(U-\frac{1}{2}\sqrt{2})^2+\frac{1}{2}}$$

$$= \tfrac{1}{4}\sqrt{2}\int\frac{(2U+\sqrt{2})\,dU}{U^2+U\sqrt{2}+1} + \tfrac{1}{2}\int\frac{dU}{(U+\frac{1}{2}\sqrt{2})^2+\frac{1}{2}}$$

$$= \tfrac{1}{4}\sqrt{2}\ln\frac{U^2-U\sqrt{2}+1}{U^2+U\sqrt{2}+1} + \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(U\sqrt{2}-1)$$

$$+ \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(U\sqrt{2}+1) + C.$$

$$= \tfrac{1}{4}\sqrt{2}\ln\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1} + \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(\sqrt{2x}-1)$$

$$+ \tfrac{1}{2}\sqrt{2}\operatorname{arc\,tg}(\sqrt{2x}+1) + C.$$

Hasil seluruhnya , dari :

$$\int\frac{1}{\sqrt{x}}\operatorname{arc\,tg}x\,dx = 2\sqrt{x}\operatorname{arc\,tg}x - \tfrac{1}{2}\sqrt{2}\ln\left|\frac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}\right|$$

$$- \sqrt{2}\operatorname{arc\,tg}(\sqrt{2x}-1) - \sqrt{2}\operatorname{arc\,tg}$$

$$(\sqrt{2}x+1) + C.$$

41). Tentukan integral :

a). $\int\frac{dx}{(a^2-x^2)^{3/2}}$ dan b). $\int\frac{\operatorname{arc\,sin}\frac{x}{a}}{(a^2-x^2)^{3/2}}dx$ (ujian th. 1939)

Penyelesaian :

a). $\int \frac{dx}{(a^2 - x^2)^{3/2}}$, untuk integral ini, dimisalkan :

$x = a \sin U$ dan $dx = a \cos U \, dU$

Setelah dimasukkan , integral menjadi :

$$\int \frac{a \cos U \, dU}{(a^2 - a^2 \sin^2 U)^{3/2}} = \int \frac{a \cos U \, dU}{a^3 \cos^3 U} = \frac{1}{a^2} \int \sec^2 U \, dU$$

$$= \frac{1}{a^2} \operatorname{tg} U + C.$$

J a d i :

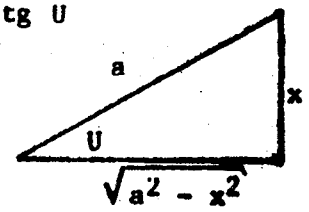
$$\int \frac{dx}{(a^2 - x^2)^{3/2}} = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C$$

b). $\int \frac{\arcsin \frac{x}{a}}{(a^2 - x^2)^{3/2}} \, dx$, diambil substitusi : $x = a \sin U$

$dx = a \cos U \, dU$, maka integral menjadi :

$$\int \frac{U \cdot a \cos U \, dU}{a^3 \cos^2 U} = \frac{1}{a^2} \int U \sec^2 U \, dU = \frac{1}{a^2} U \operatorname{tg} U$$

$$- \frac{1}{a^2} \int \operatorname{tg} U \, dU$$



$$= \frac{1}{a^2} \left[U \operatorname{tg} U + \ln \cos U \right] + C.$$

J a d i :

$$\int \frac{\arcsin \frac{x}{a}}{(a^2 - x^2)^{3/2}} \, dx = \frac{x \arcsin \frac{x}{a}}{a^2 \sqrt{a^2 - x^2}} + \frac{1}{a^2} \ln \frac{\sqrt{a^2 - x^2}}{a} + C$$

atau :

$$= \frac{x \arcsin \frac{x}{a}}{a^2 \sqrt{a^2 - x^2}} + \frac{1}{2a^2} \ln (a^2 - x^2) + C.$$

42). Tentukan : $\int x \sin x \cos x \, dx$ (ujian th. 1940).

Penyelesaian :

$$\int x \sin x \cos x \, dx = \int x \sin x \, d(\sin x)$$

$$= x \sin x \sin x - \int \sin x (\sin x + x \cos x) \, dx$$

$$= x \sin x \sin x - \int \sin x \sin x \, dx - \int x \cos x \sin x \, dx$$

$$= x \sin x \sin x - \int \sin x \sin x \, dx - x \cos x \cos x + \int \cos x (\cos x - x \sin x) \, dx$$

$$= x \sin x \sin x - x \cos x \cos x - \int \sin x \sin x \, dx + \int \cos x \cos x \, dx - \int x \sin x \cos x \, dx$$

$$2 \int x \sin x \cos x \, dx = x (\sin x \sin x - \cos x \cos x) + \int \cos x \cos x \, dx - \int \sin x \sin x \, dx$$

$$= x (\sin x \sin x - \cos x \cos x) + \cos x \sin x + \int \sin x \sin x \, dx - \int \sin x \sin x \, dx$$

Maka :

$$\int x \sin x \cos x \, dx = \tfrac{1}{2} x (\sin x \sin x - \cos x \cos x) + \tfrac{1}{2} \cos x \sin x + C.$$

43). Tentukan integral : $\int \frac{x^3 + x^2 + 1}{x^4 - 2x^3 + 3x^2 - 4x + 2} \, dx$ (1940)

Penyelesaian :

$$\int \frac{x^3 + x^2 + 1}{x^4 - 2x^3 + 3x^2 - 4x + 2} \, dx = \int \frac{x^3 + x^2 + 1}{(x - 1)^2 (x^2 + 2)} \, dx$$

Pandang dulu bentuk :

$$\frac{x^3 + x^2 + 1}{(x-1)^2 (x^2+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C\,x + D}{x^2 - 2}$$

$$x^3 + x^2 + 1 = A\,(x-1)\,(x^2+2) + B\,(x^2+2) + (Cx + D)\,(x-1)^2$$

untuk $x = 1$, $3 = 3\,B$ dan $B = 1$

$$x^3 + x^2 + 1 - x^2 - 2 = (x-1)\left[A\,(x^2+2) + (Cx+D)\,(x-1)\right]$$

$$(x-1)\,(x^2 + x + 1) = (x-1)\left[A\,(x^2+2) + (Cx+D)\,(x-1)\right]$$

$$x^2 + x + 1 = A\,(x^2+2) + (Cx+D)\,(x-1)$$

Untuk : $x = 1$, $3 = 3\,A$ dan $A = 1$

Dengan identiteit :

koeffisien x^2 : $A + C = 1$ dan $C = 0$

koeffisien x^1 : $-C + D = 1$ dan $D = 1$

M a k a :

$$\int \frac{x^3 + x^2 + 1}{x^4 - 2x^3 + 3x^2 - 4x + 2}\,dx = \int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} + \int \frac{dx}{x^2+2}$$

$$= \ln (x-1) - \frac{1}{x-1} + \tfrac{1}{2}\sqrt{2}\ \text{arc tg}\ \frac{x}{\sqrt{2}} + C.$$

44). Tentukan : $\int \frac{dx}{(4x-3)\sqrt{x^2+1}}$ (ujian tahun 1941)

Penyelesaian :

Untuk menyelesaikan integral diatas, dimisalkan :

$$4x - 3 = \frac{1}{U} \quad \text{dan} \quad x = \frac{3U+1}{4U}\ , \quad x^2 + 1 = \left(\frac{3U+1}{4U}\right)^2 + 1$$

$$dx = -\frac{dU}{4U^2}\ , \qquad = \frac{25U^2 + 6U + 1}{16U^2}$$

Maka integral menjadi :

$$\int \frac{-\frac{dU}{4\,U^2}}{\frac{1}{U}\sqrt{\frac{25\,U^2 + 6\,U + 1}{16\,U^2}}} = -\int \frac{dU}{\sqrt{25\,U^2 + 6U + 1}}$$

$$= -\frac{1}{5}\int \frac{dU}{\sqrt{(U + \frac{3}{25})^2 + \frac{16}{(25)^2}}}$$

$$= -\frac{1}{5}\ln\left| U + \frac{3}{25} + \frac{1}{5}\sqrt{25\,U^2 + 6\,U + 1}\right| + C$$

Jadi hasil integral :

$$\int \frac{dx}{(4\,x - 3)\sqrt{x^2 + 1}} = -\frac{1}{5}\ln\left| \frac{1}{4x - 3} + \frac{3}{25} + \frac{1}{5}\sqrt{25\,\frac{1}{(4\,x - 3)^2} + \frac{6}{4\,x - 1} + 1}\right|$$

$$= -\frac{1}{5}\ln\left|\frac{4\,(4 + 3x) + 20\sqrt{x^2 + 1}}{25\,(4\,x - 3)}\right| + C$$

$$= -\frac{1}{5}\ln\left|\frac{3\,x + 4 + 5\sqrt{x^2 + 1}}{4\,x - 3}\right| + C$$

45). Tentukan : $\int \frac{dx}{\sqrt{x^2 - 2\,x\cos\phi - \sin^2\phi}}$ (ujian th. 1941)

Penyelesaian :

Oleh karena integral dilaksanakan ke - x , maka ϕ = konstan.

$$\int \frac{dx}{\sqrt{x^2 - 2\,x\cos\phi - \sin^2\phi}} = \int \frac{dx}{\sqrt{x^2 - 2\,x\cos\phi + \cos^2\phi - 1}}$$

$$= \int \frac{dx}{\sqrt{(x - \cos\phi)^2 - 1}}$$

$$= \ln (x - \cos \phi + \sqrt{x^2 - 2x \cos \phi - \sin^2 \phi}) + C$$

46). Tentukan : $\int (2x + 1 - \sqrt{x^2 + x + 1})\, dx$ (ujian th. 1942)

Penyelesaian :

$$\int (2x + 1 - \sqrt{x^2 + x + 1})\, dx = \int \left[(2x + 1) - (\sqrt{x^2 + x + 1})\right] dx$$

$$= \int (2x + 1)\, dx - \int \sqrt{x^2 + x + 1}\, dx$$

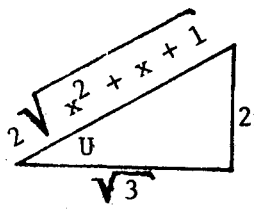
$$= x^2 + x - \int \sqrt{(x + \tfrac{1}{2})^2 + (\tfrac{1}{2}\sqrt{3})^2}\, dx$$

Untuk integral yang terakhir , dimisalkan :

$x + \frac{1}{2} = \frac{1}{2}\sqrt{3}\ \text{tg}\ U$ dan $dx = \frac{1}{2}\sqrt{3}\ \sec^2 U\, dU$, maka :

$$\int \sqrt{(x + \tfrac{1}{2})^2 + (\tfrac{1}{2}\sqrt{3})^2}\, dx = \int \tfrac{1}{2}\sqrt{3} \sec U \cdot \tfrac{1}{2}\sqrt{3} \sec^2 U\, dU$$

$$= \frac{3}{4} \sec^2 U\, dU = \frac{3}{8} \sec U \text{ tg } U + \ln (\sec U + \text{tg } U) + C$$



$$= \tfrac{1}{4}(2x + 1)\sqrt{x^2 + x + 1} + \frac{3}{8} \ln \left| \frac{2x + 1 + \sqrt{x^2 + x + 1}}{\sqrt{3}} \right| + C$$

Jadi:

$$\int (2x + 1 - \sqrt{x^2 - x + 1})\, dx = x^2 + x - \tfrac{1}{4}(2x + 1)\sqrt{x^2 + x + 1}$$

$$- \frac{3}{8} \ln (2x + 1 + \sqrt{x^2 + x + 1}) + C$$

47). Tentukan : $\int \frac{dx}{x^4 + x^3 + x + 1}$ (ujian tahun 1941)

Penyelesaian :

$$\int \frac{dx}{x^4 + x^3 + x + 1} = \int \frac{dx}{(x + 1)^2 (x^2 - x + 1)}$$

Pandang dulu bentuk :

$$\frac{1}{(x+1)^2\,(x^2-x+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1}$$

$$1 = A\,(x+1)\,(x^2-x+1) + B\,(x^2-x+1) + (Cx+D)\,(x+1)^2$$

Untuk : $x = -1$, $1 = 3B$ dan $B = \frac{1}{3}$

$$1 - \frac{1}{3}(x^2-x+1) = (x+1)\;A\,(x^2-x+1) + (Cx+D)\,(x+1)$$

$$-\frac{1}{3}(x-2) = A\,(x^2-x+1) + (Cx+D)\,(x+1)$$

untuk : $x = -1$, $1 = 3A$ dan $A = \frac{1}{3}$

Dengan identiteit :

$A + C = 0$, maka $C = -\frac{1}{3}$; $A + D = \frac{2}{3}$ dan $D = \frac{1}{3}$

J a d i :

$$\int \frac{dx}{x^4+x^3+x+1} = \frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{(x+1)^2} - \frac{1}{3}\int \frac{(x-1)\,dx}{x^2-x+1}$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{3(x+1)^2} -$$

$$\frac{1}{6}\ln(x^2-x+1) + \frac{1}{6}\int \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}}$$

$$= -\ln\left|\frac{(x+1)^2}{x^2-x+1}\right| - \frac{1}{3(x+1)} + \frac{1}{9}\,\text{arc tg}\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

48). Tentukan : $\int \frac{dx}{(x+1)\sqrt{2x^2+x-1}}$ (ujian th. 1941)

Penyelesaian :

Untuk menyelesaikan integral tersebut diatas , diambil substitusi :

$x + 1 = \frac{1}{U}$ dan $x = \frac{1 - U}{U}$; $dx = -\frac{dU}{U^2}$, sedang

$$2x^2 + x - 1 = 2\left(\frac{1-U}{U}\right) + \left(\frac{1-U}{U}\right) - 1 = \frac{2 - 3U}{U^2}$$

M a k a :

$$\int \frac{dx}{(x+1)\sqrt{2x^2 + x + 1}} = \int \frac{-\frac{dU}{U^2}}{\frac{1}{U}\sqrt{\frac{2-3U}{U^2}}} = -\int \frac{dU}{\sqrt{2 - 3U}}$$

$$= \frac{2}{3}\sqrt{2 - 3U} + C$$

$$= \frac{2}{3}\sqrt{\frac{2x - 1}{x + 1}} + C$$

49). Tentukan : $\int \frac{dx}{(x+1)\sqrt{1 - x^2}}$ (ujian tahun 1941)

Penyelesaian :

Misalkan : $1 - x^2 = (1 - x)^2\ U^2$, $1 + x = (1 - x)\ U^2$

$$x = \frac{U^2 - 1}{U^2 + 1}, \quad dx = \frac{4\ U\ dU}{(U^2 + 1)^2}$$

M a k a :

$$\int \frac{dx}{(x+1)\sqrt{1 - x^2}} = \int \frac{\frac{4\ U\ dU}{(U^2+1)^2}}{\left(\frac{2\ U^2}{U^2+1}\right)\cdot\left(\frac{2\ U}{U^2+1}\right)} = \int \frac{dU}{U^2} = -\frac{1}{U} + C.$$

Jadi :

$$\int \frac{dx}{(1+x)\sqrt{1 - x^2}} = -\sqrt{\frac{1-x}{1+x}} + C$$

50). Tentukan : $\int \frac{dx}{(1 + e^{-x})^{3/4}}$ (ujian tahun 1942)

Penyelesaian :

Andaikan : $1 + e^{-x} = U^4$, $- e^{-x} dx = 4 U^2 dU$ dan

$$dx = - \frac{4 U^2 \, dU}{U^4 - 1}$$

Maka integral :

$$\int \frac{dx}{(1 + e^{-x})^{3/4}} = \int - \frac{\frac{4 U^3 \, dU}{U^4 - 1}}{U^3} = - 4 \int \frac{dU}{(U - 1)(U + 1)(U^2 + 1)}$$

Pandang dulu bentuk :

$$\frac{4}{(U - 1)(U + 1)(U^2 + 1)} = \frac{A}{U - 1} + \frac{B}{U + 1} + \frac{C U + D}{U^2 + 1}$$

$$4 = A (U + 1)(U^2 + 1) + B (U - 1)(U^2 + 1) + (CU + D)(U - 1)(U + 1)$$

Untuk U = 1 , $4 = 4 A$ dan $A = 1$

U = - 1 , $4 = -4 B$ dan $B = - 1$

Dengan identiteit :

koeffisien U^3 : $A + B + C = 0$ dan $C = 0$

koeffisien U^0 : $A - B - D = 4$ dan $D = - 2$

Maka :

$$\int \frac{4 \, dU}{(U - 1)(U + 1)(U^2 + 1)} = \int \frac{dU}{U - 1} - \int \frac{dU}{U + 1} - 2 \int \frac{dU}{U^2 + 1}$$

$$= \ln (U - 1) - \ln (U + 1) - 2 \text{ arc tg } U + C.$$

Jika U diganti dengan $\sqrt[4]{1 + e^{-x}}$, maka terdapat :

$$\int \frac{dx}{(1+e^{-x})^{3/4}} = -\ln\left(\sqrt[4]{1+e^{-x}} - 1\right) + \ln\left(\sqrt[4]{1+e^{-x}} + 1\right)$$

$$+ 2 \arctan\left(\sqrt[4]{1+e^{-x}}\right) + C$$

$$= \ln\left|\frac{\sqrt[4]{1+e^{x}} + \sqrt[4]{e^{x}}}{\sqrt[4]{1+e^{x}} - \sqrt[4]{e^{x}}}\right| + 2 \arctan \sqrt[4]{1+e^{-x}} + C$$

51). Tentukan integral : $\int \frac{x^3}{\sqrt{2x - x^2}}\, dx$ (ujian th. 1942)

Penyelesaian :

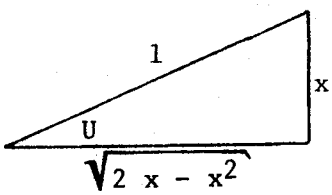
$$\int \frac{x^3}{\sqrt{2x - x^2}}\, dx = \int \frac{x^3\, dx}{\sqrt{1 - (x-1)^2}}$$

Misalkan : $x - 1 = \sin U$, $x = 1 + \sin U$ dan $dx = \cos U\, dU$

Maka integral menjadi :

$$\int \frac{(1+\sin U)^3 \cos U\, dU}{\cos U} = \int (1 + 3\sin U + 3\sin^2 U + \sin^3 U)\, dU$$

$$= U - 3\cos U + \frac{3}{2}\int (1 - \cos 2U)\, dU - \int (1 - \cos^2 U)\, d\cos U$$



$$= U - 3\cos U + \frac{3}{2}(U - \tfrac{1}{2}\sin 2U) - \cos U + \frac{1}{3}\cos^3 U + C$$

$$= \frac{5}{2}U - 4\cos U - \frac{3}{4}\sin 2U + \frac{1}{3}\cos^3 U + C$$

kemudian u diganti , jadi :

$$\int \frac{x^3}{2x - x^2}\, dx = \frac{5}{2}\arcsin(x-1) - 4\sqrt{2x - x^2} - \frac{3}{2}(x-1)$$

$$\sqrt{2x - x^2} + \frac{1}{3}\sqrt{(2x - x^2)^3} + C.$$

52). Hitung integral : $\int \frac{x-1}{\sqrt{2x-x^2}} \ln \frac{1-x}{\sqrt{2x-x^2}}\, dx$ (ujian th.1942)

Penyelesaian :

$$\int \frac{x-1}{\sqrt{2x-x^2}} \ln \frac{1-x}{\sqrt{2x-x^2}}\, dx = -\int \ln \frac{1-x}{\sqrt{2x-x^2}}\, d\left(\sqrt{2x-x^2}\right)$$

$$= -\sqrt{2x-x^2} \ln \frac{1-x}{\sqrt{2x-x^2}} + \int \sqrt{2x-x^2}\, d\left(\ln \frac{1-x}{\sqrt{2x-x^2}}\right)$$

$$= -\sqrt{2x-x^2} \ln \frac{1-x}{\sqrt{2x-x^2}} - \int \frac{dx}{(1-x)\sqrt{2x-x^2}}$$

Pandang dulu bentuk :

$$\int \frac{dx}{(1-x)\sqrt{2x-x^2}} = \int \frac{dx}{(1-x)\sqrt{1-(x-1)^2}}$$

Misalkan : $x - 1 = \sin U$ dan $dx = \cos U\, dU$, maka

Integral menjadi :

$$-\int \frac{\cos U\, dU}{\sin U \cos U} = -\int \operatorname{cosec} U\, dU = -\ln \left| \operatorname{cosec} U - \operatorname{ctg} U \right| + C$$

Maka :

$$\int \frac{dx}{(1-x)\sqrt{2x-x^2}} = -\ln \left| \frac{1-\sqrt{2x-x^2}}{x-1} \right| + C$$

Jadi keseluruhan :

$$\int \frac{x-1}{\sqrt{2x-x^2}} \ln \frac{1-x}{\sqrt{2x-x}}\, dx = -\sqrt{2x-x^2} \ln$$

$$\left| \frac{1-x}{\sqrt{2x-x^2}} \right| + \ln \left| \frac{1-\sqrt{2x-x^2}}{x-1} \right| + C$$

53). Hitung integral : $\int \frac{x \ln x}{\sqrt{x^2 + 1}} \, dx$ (ujian th. 1942)

Penyelesaian :

$$\int \frac{x \ln x}{\sqrt{x^2 + 1}} \, dx = \int \ln x \, d\left(\sqrt{x^2 + 1}\right) = \sqrt{x^2 + 1} \ln x - \int \frac{\sqrt{x^2 + 1}}{x} \, dx$$

Pandang dulu bentuk : $\int \frac{\sqrt{x^2 + 1}}{x} \, dx$

Misalkan : $x^2 + 1 = U^2$ dan $2 x \, dx = 2 U \, dU$, $\frac{dx}{x} = \frac{U \, dU}{U^2 - 1}$

Maka : $\int \frac{\sqrt{x^2 + 1}}{x} \, dx = \int \frac{U^2 \, dU}{U^2 - 1} = \int dU + \int \frac{dU}{U^2 - 1}$

$$= U + \tfrac{1}{2} \ln \left| \frac{U - 1}{U + 1} \right| + C$$

$$= x^2 + 1 + \tfrac{1}{2} \ln \left| \frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 1} + 1} \right| + C$$

Jadi, hasil seluruhnya :

$$\int \frac{x \ln x}{\sqrt{x^2 + 1}} \, dx = \sqrt{x^2 + 1} \, (\ln x - 1) - \tfrac{1}{2} \ln \left| \frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 1} + 1} \right| + C$$

54). Hitung integral : $\int \frac{x^3 \arcsin x}{\sqrt{x - x^4}} \, dx$ (ujian th. 1943)

Penyelesaian :

$$\int \frac{x^3 \arcsin x}{\sqrt{1 - x^4}} \, dx = - \tfrac{1}{2} \int \arcsin x \, d\left(\sqrt{1 - x^4}\right)$$

$$= - \tfrac{1}{2} (\arcsin x) \sqrt{1 - x^4} + \tfrac{1}{2} \int \frac{\sqrt{1 - x^4}}{\sqrt{1 - x^2}} + dx$$

$$= -\tfrac{1}{2} (\arcsin x) \sqrt{1 - x^4} + \tfrac{1}{2} \int \sqrt{1 + x^2}\, dx$$

$$= -\tfrac{1}{2} (\arcsin x) \sqrt{1 - x^4} + \tfrac{1}{4} x \sqrt{x^2 + 1}$$

$$+ \tfrac{1}{4} \ln (x + \sqrt{x^2 + 1}) + C$$

55). Tentukan : $\int \sqrt{x + \sqrt{x}}\, dx$ (ujian th 1945)

Penyelesaian :

$$\int \sqrt{x + \sqrt{x}}\, dx = \int \sqrt{(\sqrt{x} + \tfrac{1}{2})^2 - \tfrac{1}{4}}\, dx$$

Misalkan : $\sqrt{x} + \tfrac{1}{2} = \tfrac{1}{2} \sec U$, $\sqrt{x} = \tfrac{1}{2} (\sec U - 1)$

dan $\frac{dx}{2\sqrt{x}} = \tfrac{1}{2} \sec U \operatorname{tg} U\, dU$, $dx = \tfrac{1}{2} \sec U \operatorname{tg} U (\sec U - 1)\, dU$

Maka bentuk integral menjadi :

$$\int \sqrt{\tfrac{1}{4} \operatorname{tg}^2 U} \cdot \tfrac{1}{2} \sec U \operatorname{tg} U (\sec U - 1)\, dU$$

$$= \tfrac{1}{4} \int \operatorname{tg}^2 U (\sec^2 U - \sec U)\, dU$$

$$= \tfrac{1}{4} \int \operatorname{tg}^2 U \, d\operatorname{tg} U - \tfrac{1}{4} \int (\sec^2 U - 1) \sec U\, dU$$

$$= \frac{1}{12} \operatorname{tg}^3 U - \frac{1}{8} \sec U \operatorname{tg} U + \frac{1}{8} \ln(\sec U + \operatorname{tg} U) + C$$

Jadi :

$$\int \sqrt{x + \sqrt{x}}\, dx = \frac{2}{3} (\sqrt{x + \sqrt{x}})^3 - \tfrac{1}{4} (2\sqrt{x} + 1) \sqrt{x + \sqrt{x}}$$

$$+ \frac{1}{8} \ln \left| 2\sqrt{x} + 1 + 2\sqrt{x + \sqrt{x}} \right| + C.$$

56). Tentukan : $\int \frac{3x^4 + 1}{(x^4 - 1)^2}\, dx$ (ujian th. 1945)

Penyelesaian :

$$\int \frac{3x^4 + 1}{(x^4 - 1)^2}\, dx = \int \frac{3x^4 + 1}{(x + 1)^2 (x - 1)^2 (x + 1)^2}\, dx$$

Pandang dulu bentuk :

$$\frac{3x^4 + 1}{(x+1)^2 (x-1)^2 (x^2+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

$$\frac{D}{(x-1)^2} + \frac{Ex + F}{x^2 + 1} + \frac{Gx + H}{(x^2+1)^2}$$

$$3x^4 + 1 = A(x+1)(x-1)^2(x^2+1)^2 + B(x-1)^2(x^2+1)^2$$

$$+ C(x+1)^2(x-1)(x^2+1)^2 + D(x+1)^2(x^2+1)^2$$

$$+ (Ex + F)(x+1)^2(x-1)^2(x^2+1)$$

$$+ (Gx + H)(x+1)^2(x-1)^2$$

Untuk : $x = 1$, $4 = 16D$ dan $D = \frac{1}{4}$

$x = -1$, $4 = 16D$ dan $B = \frac{1}{4}$

Dimasukkan harga-harga ini kedalam persamaan , dan kemudian disederhanakan , terdapat :

$$3x^4 + 1 - \tfrac{1}{4}(x-1)^2(x^2+1)^2 - \tfrac{1}{4}(x+1)^2(x^2+1)^2$$

$$= (x+1)(x-1)\,A(x-1)(x^2+1)^2 + C(x+1)(x^2+1)^2$$

$$+ (Ex + F)(x+1)(x-1)(x^2+1)$$

$$+ (Gx + H)(x+1)(x-1) \quad .$$

$$-\tfrac{1}{2}(x-1)^2(x+1)^2 = A(x-1)(x^2+1)^2 + C(x-1)(x^2+1)^2$$

$$+ (Ex + F)(x+1)(x-1)(x^2+1)$$

$$+ (Gx + H)(x+1)(x-1)$$

Untuk : $x = 1$, $0 = -8A$ dan $A = 0$

$x = -1$, $0 = 8C$ dan $C = 0$

$$-\tfrac{1}{2}(x^2 - 1) = (Ex + F)(x^2 + 1) + (Gx + H)$$

Terdapat : E = 1 , F = - ½ , G = 0 dan H = 1.

Maka :

$$\int \frac{3x^4 + 1}{(x^4 - 1)^2}\,dx = \tfrac{1}{4}\int \frac{dx}{(x+1)^2} + \tfrac{1}{4}\int \frac{dx}{(x-1)^2} - \tfrac{1}{2}\int \frac{dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$$

Pandang dulu bentuk :

$\int \frac{dx}{(x^2+1)^2}$. Andaikan : $x = \operatorname{tg} U$ dan $dx = \sec^2 U\, dU$

Maka integral menjadi :

$$= \int \frac{\sec^2 U\, dU}{\sec^4 U} = \int \cos^2 U\, dU = \tfrac{1}{2}\int (1 + \cos 2U)\, dU$$

$$= \tfrac{1}{2}u + \tfrac{1}{4}\sin 2U + C = \tfrac{1}{2}\operatorname{arc\,tg}\, x + \frac{x}{2(x^2+1)} + C$$

Hasil integral yang terakhir :

$$\int \frac{3x^4 + 1}{(x^4 - 1)^2}\,dx = \frac{x}{2(x^2+1)} - \frac{1}{4(x+1)} - \frac{1}{4(x-1)}$$

$$- \tfrac{1}{2}\operatorname{arc\,tg}\, x + \tfrac{1}{2}\operatorname{arc\,tg}\, x + C.$$

$$= \frac{x}{2(x^2+1)} - \frac{1}{4(x+1)} - \frac{1}{4(x-1)} + C$$

$$= -\frac{x}{x^4 - 1} + C$$

57). Hitung integral : $\int \frac{dx}{\cos x \sqrt{\cos 2x}}$ (ujian th. 1946)

Penyelesaian :

$$\int \frac{dx}{\cos x \sqrt{\cos 2x}} = \int \frac{dx}{\cos x \sqrt{\cos^2 x - \sin^2 x}}$$

$$= \int \frac{\sec^2 x}{\sqrt{1 - tg^2 x}}\, dx = \text{arc sin } (tg\ x) + C$$

58). Tentukan : $\int \frac{\sqrt{x + \sqrt{x^2 + 1}}}{\sqrt{x^2 + 1}}\, dx$ (ujian th. 1946).

Penyelesaian :

Untuk menyelesaikan integral diatas , dimisalkan :

$x = tg\ U$ dan $dx = \sec^2 U\, dU$; sedang $x^2 + 1 = tg^2 U + 1 = \sec^2 U$

maka :

$$\int \frac{\sqrt{x + \sqrt{x^2 + 1}}}{\sqrt{x^2 + 1}}\, dx = \int \frac{\sqrt{tg\ u + \sec\ U}}{\sec U}. \ (\sec^2 U\, dU)$$

$$= \int \sqrt{tg\ U + \sec\ U}\ .\ \sec U\, dU$$

$$= \int \sqrt{tg\ U + \sec\ U}\ .\ \frac{\sec U\ (\sec U + tg\ U)}{\sec U + tg\ U} dU$$

$$= \int \frac{d\ (\sec U + tg\ U)}{\sqrt{\sec U + tg\ U}}$$

$$= 2\sqrt{\sec U + tg\ U} + C$$

Jadi : $\int \frac{\sqrt{x + \sqrt{x^2 + 1}}}{\sqrt{x^2 + 1}}\, dx = 2\sqrt{x + \sqrt{x^2 + 1}} + C$

59). Hitung integral : $\int \sqrt{1 - x^2}\ \text{arc sin}\ \sqrt{1 - x^2}\, dx$ (ujian th 1946)

Penyelesaian :

Untuk integral tersebut diatas diambil :

$\text{arc sin}\ \sqrt{1 - x^2} = U$, $\sqrt{1 - x^2} = \sin U$ atau $x^2 = 1 - \sin^2 U$

$x = \cos U$, dan $dx = -\sin U\, dU$

M a k a :

$$\int \sqrt{1 - x^2}\ \text{arc sin}\sqrt{1 - x^2}\ dx = \int \sin U \,.\, U \,.\, (-\sin U\, dU)$$

$$= -\int U \sin^2 U\, dU = -\tfrac{1}{2}\int U\,(1 - \cos 2U)\, dU$$

$$= -\tfrac{1}{4} U^2 + \tfrac{1}{4} U \sin 2U + \frac{1}{8}\cos 2U + C$$

$$= -\tfrac{1}{4}\,(\text{arc sin}\sqrt{1 - x^2})^2 + \tfrac{1}{2} x \sqrt{1 - x^2}\ \text{arc sin}$$

$$\sqrt{1 - x^2} + \frac{1}{8}\,(2x^2 - 1) + C.$$

60). Tentukan : $\int \dfrac{\ln x}{(x+1)\sqrt{x+1}}\, dx$ (ujian th. 1947)

Penyelesaian :

$$\int \frac{\ln x}{(x+1)\sqrt{x+1}}\, dx = \int \frac{\ln x}{(x+1)^{3/2}}\, dx = -2\int \ln x\, d\,(x+1)^{-\frac{1}{2}}$$

$$= -\frac{2\ln x}{\sqrt{x+1}} + 2\int \frac{dx}{x\sqrt{x+1}}$$

Pandang dulu bentuk :

$\int \dfrac{dx}{x\sqrt{x+1}}$. Misalkan : $x + 1 = U^2$ dan $x = U^2 - 1$

$dx = 2\,U\,dU$

Maka bentuk integral menjadi :

$$\int \frac{2\,U\,dU}{(U^2 - 1)\,U} = 2\int \frac{dU}{U^2 - 1} = \ln\left|\frac{U - 1}{U + 1}\right| + C$$

$$= \ln\left|\frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}\right| + C$$

Jadi :

$$\int \frac{\ln x}{(x+1)\sqrt{x+1}}\, dx = -\frac{2\ln x}{\sqrt{x+1}} + 2\ln\left|\frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1}\right| + C$$

61). Tentukan : a). $\int \dfrac{\sin 2x}{1 + \cos x}\, dx$ dan b). $\int \dfrac{(1 + x)e^x}{1 + x^2 e^{2x}}$ (th. 1947).

Penyelesaian :

a). $\int \frac{\sin 2x}{1 + \cos x}\, dx = 2\int \frac{\cos x \sin x}{1 + \cos x} dx = -2\int \frac{\cos x}{1 + \cos x}\, d(\cos x).$

$$= -2\int \frac{\cos x + 1 - 1}{1 + \cos x}\, d(\cos x)$$

$$= 2\int \frac{d(\cos x)}{1 + \cos x} - 2\int d(\cos x)$$

$$= 2 \ln (1 + \cos x) - 2 \cos x + C.$$

b). $\int \frac{(1 + x)\, e^x}{1 + x^2\, e^{2x}}\, dx$. Misalkan : $x\, e^x = U$, $(1 + x)\, e^x\, dx = dU$

maka integral menjadi :

$$= \int \frac{du}{1 + U^2} = \text{arc tg } U + C = \text{arc tg } (x\, e^x) + C$$

Jadi : $\int \frac{(1 + x)\, e^x}{1 + x^2\, e^{2x}}\, dx = \text{arc tg } (x\, e^x) + C.$

62). Tentukan : $\int \frac{dx}{\sqrt{2x - 1} - \sqrt[4]{2x - 1}}$

Penyelesaian :

Untuk menyelesaikan integral tersebut , dimisalkan :

$2x - 1 = U^4$ dan $dx = 2\, U^3\, dU$. Maka integral menjadi :

$$\int \frac{2\, U^3\, dU}{U^2 - U} = 2\int \frac{U^2}{U - 1}\, dU = 2\int \left(U + 1 + \frac{1}{U - 1}\right) dU$$

$$= U^2 + 2U + 2 \ln (U - 1) + C$$

$$= \sqrt{2x - 1} + 2\sqrt[4]{2x - 1} + 2 \ln \left(\sqrt[4]{2x - 1} - 1\right) + C$$

Jadi :

$$\int \frac{dx}{\sqrt{2x-1} - \sqrt[4]{2x-1}} = \sqrt{2x-1} + 2\sqrt[4]{2x-1} +$$

$$+ 2 \ln\left(\sqrt[4]{2x-1} - 1\right) + C .$$

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